

The LAIMA series



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EVENTS IN  
MATHEMATICS

Part II

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This book is based on lectures given to students – future mathematics teachers – during the period of almost 30 years. It covers the period from the invention of the calculus to the end of the 20<sup>th</sup> century.

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## Preface

The second volume of this little history traces events in and to some extent around mathematics in the period starting with the giants Newton and Leibniz. The selection among the multitude of possible topics is rather standard, with the exception of mathematics in Finland, which is exposed in much more detail than its relative importance would require. The reader may regard the treatment of Finland here as a case study of a small nation in the periphery of the scientific world.

As in the first volume, I do not indicate my sources, nor do I list the literature which I have consulted. In defence of this omission I can refer to the general expository level of the presentation. Those willing to gain a deeper and more detailed insight in the various aspects of the history of mathematics will easily find a multitude of literary and internet sources to consult.

*Oulu, Finland, April 2009*  
*Matti Lehtinen*

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## About the LAIMA series

In 1990, the international team competition "Baltic Way" was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million people stood hand in hand along the Tallinn – Riga – Vilnius road, demonstrating their will for freedom.

Today "Baltic Way" has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country in the world which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The "Baltic Way" competition has given rise to other mathematical activities, too. One of them is the project LAIMA (Latvian–Icelandic Mathematics Project). Its aim is to publish a series of books covering all essential topics in the arena of mathematical competitions.

Mathematical olympiads today have become an important and essential part of the education system. In some sense they provide high standards for teaching mathematics on an advanced level. Many outstanding scientists are involved in composing problems for competitions. The "olympiad curriculum", considered all over the world, is a good reflection of important mathematical ideas on elementary level.

It is the opinion of the publishers of the LAIMA series that there are relatively few important topics which cover almost everything that the international mathematical community has recognized as worthy to be included regularly in the

search and promotion of young talent. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within the LAIMA project.

30 books have been published so far in Latvian. As LAIMA is rather a process than a project, there is no idea of a final date; many of the already published teaching aids are second or third versions and they will be extended regularly.

Benedict Johannesson, President of the Icelandic Society of Mathematics, gave inspiration to the LAIMA project in 1996. Being a co-author of many LAIMA publications, he also was the main sponsor for many years.

This book is the tenth LAIMA publication in English.

## 8 Newton and Leibniz

Many of the central ideas of differential and integral calculus already existed around 1650. But it was not before the discoveries of *Isaac Newton* (1642–1727) and *Gottfried Wilhelm Leibniz* (1646–1716) that the fundamental interconnectedness of the problems of determining areas and determining tangents was understood and the more or less isolated *ad hoc* tricks mainly based on Euclidean geometry were developed into a unified method, the *Calculus*.

The invention of the Calculus has raised different and sometimes heated opinions have been expressed on the originality or possible plagiarism. The discussion mainly concerned Leibniz, but also insinuations against Newton came up. The lack of any further evidence indicates that it is safest to assume that Newton and Leibniz constructed their systems independently of each other. Both men built on existing knowledge. The calculus of Newton and Leibniz was not indisputably rigorous in the style of Euclidean geometry. Rigor got its entry to mathematical analysis only in the 19th century. But there was no shortage of results and applications already before that.

### 8.1 The binomial series

As was the custom, Newton received an education in classical languages. He only became interested in mathematics when he was studying in Cambridge. Mathematics was taught there by Isaac Barrow. Newton made all his greatest scientific discoveries, the *binomial series*, *differential and*

*integral calculus* and the *universal law of gravitation* in 1665 and 1666 when he still was a student and the university was closed because of a plague. At old age he wrote:

*"All this was in the two plague years 1665 & 1666 for in those days I was in the prime of my age for invention and minded Mathematics and Philosophy more than at any time since."*

The central tool in Newton's mathematics was the binomial series, in modern notation

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2}x^2 + \frac{r(r-1)(r-2)}{6}x^3 + \dots$$

If  $r$  is a positive integer, the binomial series is the binomial formula, well known at Newton's time, and essentially equivalent to Pascal's triangle. But Newton did not discover the series by a generalization of the binomial formula. In fact, Newton investigated the areas under the curve  $y = (1-x^2)^{n/2}$  which had been determined by Wallis. For even  $n$  they are polynomials in  $x$ . The areas for odd  $n$  were obtained by Wallis by a certain interpolation. Newton observed that the values of  $(1-x^2)^{n/2}$  itself could be interpolated for odd  $n$  in a way which was compatible with the expressions for area. And this interpolation formula could then be generalized for arbitrary exponents.

Newton made his discovery public in 1676, in a letter to the secretary of the Royal Society. The formula then was

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DC + \text{etc.}$$

To see that this indeed is the binomial series one has to observe that the symbols  $A$ ,  $B$ ,  $C$  etc. always refer to the

previous term in the sum. The binomial series, although without any considerations of convergence, was an important new mathematical tool as it made possible the representation of functions in terms of infinite processes, not only in approximation but exactly. – Newton’s short announcement of the binomial series is the earliest instance where fractional and negative exponents appear in the way they do nowadays. The ”indices” of Wallis were similar in content, but the present notation was first used by Newton.

## 8.2 Newton’s differential and integral calculus

Newton’s first article which deals with differential and integral calculus is from 1666. *De analysi per æquationes numero terminorum infinitas* was written in 1669 and was known as a manuscript in England. The first printed version of Newton’s calculus appeared in 1704 as the appendix *De quadratura curvarum* of *The Opticks*. The final version of Newton’s calculus was *Methodus fluxionum et serierum infinitarum*, written in 1671 but only published posthumously in 1736.

The basis of Newton’s differential calculus was a physical analogy: a curve can be thought to arise when the motions of two points moving on two axes are combined. If  $x$  and  $y$  are functions of time, *fluents* in Newton’s vocabulary, they will increase in a short interval of time  $o$  by  $po$  and  $qo$ . Then the inclination of the tangent of a curve  $f(x, y) = 0$  will be  $\frac{q}{p}$ , i.e. the ratio of the momentary changes of  $y$  and  $x$ . To determine this ratio, Newton used his binomial series. For instance to determine the inclination of the tangent of the curve  $y^n = x^m$  one applies the binomial series to the equation  $(y + oq)^n = (x + op)^m$ , divides by  $o$  and then

forgets the  $o$  terms:

$$\frac{q}{p} = \frac{m x^{m-1}}{n y^{n-1}} = \frac{m}{n} x^{\frac{m}{n}-1}.$$

The decisive point in the development of the differential and integral calculus was the observation of the inverse property of the differential and integral operations. This observation was made by Newton when he examined the area under the curve  $y = f(x)$ . If, for instance, this area is

$$z = \frac{n}{m+n} a x^{\frac{m+n}{n}}$$

and  $x$  receives the increment  $o$ , then the increment of  $z$  is  $oy$  and the equation

$$z + oy = \frac{n}{m+n} a(x+o)^{\frac{m+n}{n}}$$

yields, again by the use of the binomial series,  $y = ax^{\frac{m}{n}}$ . The analysis of functions more complicated than the power function was reduced to that of power functions by again using the binomial series. – Problems related to the convergence of the series received some attention from Newton although he did not seriously consider them.

Newton called the "time derivatives"  $p$  and  $q$  of his fluents or functions  $x$  and  $y$  *fluxions*; the notation he used was  $p = \dot{x}$ ,  $q = \dot{y}$ . In a corresponding manner, fluents whose fluxions were  $x$  or  $y$  were designated by  $\dot{x}$  and  $\dot{y}$ . This notation prevailed in Britain to the end of the 19th century, and it is still used in mechanics. The Fluxion Calculus consisted of two kinds of problems: one was to derive from relations between fluents relations between fluents and their fluxions, the other was to derive from relations between fluents and fluxions relations between fluents only. The first problem is equivalent

to differentiation while the latter means solving differential equations. Of the basic relationships in the differential calculus, the chain rule for differentiating compound functions is easily understood.

The concept of a limit, essential in the rigorous foundation of the calculus, was not quite clear to Newton. According to what is needed, "small increments" are sometimes equal to zero, and they can be forgotten, and sometimes small non-zero numbers which can be cancelled. When defining his derivative, Newton speaks of the *ultimate ratio of vanishing quantities* or the *first ratio of nascent quantities*, in places where modern mathematics considers the limit

$$\lim_{h \rightarrow 0} \frac{f(h)}{g(h)}.$$

Although the logical foundation of Newton's theory remained vague, he developed a more unified and practical differential and integral calculus than the previous considerations which had dealt with isolated cases. Newton's method was a Calculus, and it contained most of the typical details of the method like the chain rule and integration by substitution.

Newton's most important work is the monumental *Philosophiæ Naturalis Principia Mathematica* (1687), a general treatise of mechanics and gravitation. The mathematical methods of the Principia are almost entirely traditional. It has been suggested that Newton would first have derived his results using the Calculus and then transformed them to the Euclidean format. It has not been possible to verify this claim.

The *Methodus fluxionum* contains the so called Newton's method or *Newton-Raphson method* (*Joseph Raphson*, 1648–1715) for approximate solution of equations. The

method is based on the repeated replacement of the graph of a function by its tangent. Newton, however, arrived at the method via the binomial series. If the equation to be solved is  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ , and  $x_1$  is an approximation of the root, then  $f(x_1 + p) = f(x_1) + na_n p x_1^{n-1} + (n-1)a_{n-1} p x_1^{n-2} + \dots + a_1 p + \text{higher powers of } p$ , which makes the equation  $f(x_1 + p) = 0$  approximately true when  $p = -\frac{f(x_1)}{f'(x_1)}$ . Here  $f'(x)$  is the "formal derivative" of  $f(x)$  or the polynomial  $na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots$ . The number  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$  is a new approximation to the solution etc.

Newton used his approximation method to form new series by "inversion" of series. *Nicolaus Mercator* (1620–87) (he is a different person than the cartographer Gerhard Mercator or Kremer) and Newton himself had derived the formula for the area under the curve  $y(1+x) = 1$ . It was  $z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$  (or the series of  $\ln(1+x)$ ). From this Newton solved  $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots$ , thus obtaining the series for the exponential function  $e^z - 1$ . Likewise the series of the integral

$$\int_0^x (1-t^2)^{1/2} dt,$$

which Newton obtained when deriving the binomial series, immediately yields a series for the area of a circular sector whose central angle  $\theta$  satisfies  $x = \sin \theta$ ; by solving  $\theta$  from the series, Newton derived the first terms of the series for  $\sin x$  and  $\cos x$ . He then recognized these to be of the form

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots, \quad \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

Newton's contribution to the development of analytic geometry is significant, too. He produced a classification of cubic curves, which is considered to be the first genuinely new result obtained by analytic geometry – the earlier results had been new derivations of truths known already to Apollonios, among others.

Newton was professor in Cambridge for 30 years – a position given to him in 1669 when Barrow had voluntarily resigned his post in favour of his talented pupil. Later Newton assumed the directorship of the Royal Mint. Newton's work as university teacher left few permanent marks. A large part of his scientific effort was directed to attempts of transforming base metals to gold as well as theology. Newton was particularly interested in early Christianity and Aryanism, the branch of Christianity which did not suppose the three-fold essence of God.

### **8.3 Leibniz**

Leibniz, a philosopher and universal genius, was born in Leipzig. He showed an exceptional talent already at a young age: the envious professors in the University of Leipzig rejected his excellent doctoral thesis in philosophy because the author was too young, only 20 years old. (Leibniz was sure to get his doctorate next year in the University of Nuremberg, with a thesis on the teaching of law. He had written the thesis on his way from Leipzig to Nuremberg.) Leibniz worked in diplomatic and administrative duties, first in the service of the archbishop of Mainz, then under the Elector of Hannover. Besides philosophy and mathematics, Leibniz worked at least in law, politics, theology, history, geology and physics. It was estimated in 2000 that the complete version of his collected works will be ready in 2055.

As mathematician, Leibniz was self-taught and as is com-

mon with self-taught scholars, he reinvented many things that were known already before him. In the history of mathematics, Leibniz is remembered besides being co-inventor of the differential and integral calculus, also as a pioneer of symbolic logic and mechanical computing aids. He sketched a *universal calculus*, with which all problems of philosophy could be solved by calculations. Leibniz constructed mechanical calculating machines and predicted a future for them: "It does not make sense that wise men slave for hours and hours with computations which anybody could do easily with machines."

Leibniz himself has to some extent described the developments of thoughts that eventually led to the calculus. Like Newton, he got the first impulses to the calculus from infinite series. His insight was to consider terms of a series as differences of consecutive terms in some sequence of numbers. This idea e.g. enabled him to calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

This problem had been presented to Leibniz by the Dutchman *Christiaan Huygens* (1629–95), one of the most influential scientists of the 17th century. In a more general level, Leibniz developed the *harmonic triangle*, which resembles the Pascal triangle. Its uppermost row is the harmonic series and the lower rows are formed by differences of the elements on the row above:

$$\begin{array}{cccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \cdots & \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \cdots & & \\ \frac{1}{4} & \frac{1}{20} & \cdots & & & \\ \frac{1}{5} & & & & & \end{array}$$

Starting from the second row, the sum of the elements on each row is the element above the first element in the sum. A related phenomenon is the expression of the area under the graph of a function as the sum of the differences of the "adjacent" values of the integral function, and as the sum of the values of the original function. Leibniz's first steps towards the calculus were taken with "functions"  $y = y(x)$ , where  $x$  can be understood as the ordinal number of the term and " $dx$ " equals 1. These steps can be followed from Leibniz's surviving notes.

In 1673 Leibniz became acquainted with the work of Pascal on the summation of sines, mentioned earlier. This opened the general significance and usefulness of the "characteristic triangle" ( $dx, dy, ds$ ). A simple geometric consideration showed that the problem of determining area, "integration", was equivalent to the problem of constructing a curve from its known tangents. Pascal had only examined the special case of a circumference of a circle of radius  $r$ . He had used the similarity of the characteristic triangle and the triangle formed by the centre, a point on the circumference and its projection on the  $x$ -axis to derive the relation essentially equivalent to the relation

$$\int y ds = \int r dx.$$

This made the calculation of the surface area of a sphere easy. Leibniz observed that the characteristic triangle made it possible to transform "integrations" associated with any curve. In particular, if one associates with the curve  $y = f(x)$  another curve defined by  $z = y - x \frac{dy}{dx}$  (in modern notation), an easy geometric argument shows that

$$\int_a^b y dx = \frac{1}{2} \left( bf(b) - af(a) + \int_a^b z dx \right).$$

When Leibniz applied this relation, for which he coined the name transmutation, to the area of a quarter of a circle, he got the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots,$$

which is named after him. – A couple of years later Leibniz was in the possession of derivation and integration with the present-day differential notation. Originally Leibniz denoted the infinitesimal difference of two adjacent values of the variable  $y$  with the symbol  $\ell$  and the sum of the values of a variable by  $\text{omn.}\ell$ . So for instance  $\frac{1}{2}y^2 = \int y dy$  would be denoted by

$$\frac{\overline{\text{omn.}\ell}^2}{2} = \text{omn.}\overline{\text{omn.}\ell} \frac{\ell}{a}$$

( $a = 1$  is in the formula for dimensional reasons and a bar over a formula has the same meaning as brackets around it.) In a manuscript from 1675 Leibniz replaces the  $\text{omn.}$  sign with the integral sign  $\int$  and slightly later  $\ell$  is replaced by the operational symbol  $d$ . After pondering the possibility of  $d(uv) = du dv$  Leibniz observes that in any case  $d(x^2) = (x + dx)^2 - x^2 = 2x dx + (dx)^2 = 2x dx$  and comes to the correct differentiation rules. The fundamental theorem of calculus becomes clear to Leibniz after he observes that the area associated with the curve with ordinates  $z$  can be found if one find a curve such that its ordinates  $y$  satisfy  $\frac{dy}{dx} = \frac{z}{a}$  ( $a$  is again there to preserve dimensions). This makes  $z dx = a dy$ , and the area is

$$\int z dx = a \int dy = ay.$$

Leibniz usually made his curves pass the origin.

In 1684 Leibniz began the publication of notes on calculus in the journal *Acta Eruditorum Lipsienium* (Works of Leipzig Scholars) edited by him. This was the forum for most of the early communications of Leibnizian calculus. The first publication, in six pages, also contained the first physical application of the new analysis: the derivation "like with magic" of Snell's law of the refraction of light traveling from one substance to another. – In the 17th century, correspondence was almost the only way of relatively rapid communication of new results; the birth of scientific journals had a remarkable impact on the change of ideas.

The symbols invented by Leibniz were successful. Although the exact significance of  $dx$  and  $dy$  remains undefined, operation with them is intuitively clear and usually leads to correct results. The "chain rule" " $(f \circ g)'(x) = f'(g(x))g'(x)$ " is much more understandable in the form

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

and – to give another example – the derivation of the differentiation rule of a quotient in the form

$$d\frac{y}{x} = \frac{y + dy}{x + dx} - \frac{y}{x} = \frac{xy + xdy - xy - ydx}{x^2 + xdx} = \frac{xdy - ydx}{x^2}$$

is intuitively clear. Leibniz was one of the users of term such that *function*, *constant*, *variable* and *parameter*.

Reasons like these made the development of the calculus to proceed in the direction pointed by Leibniz. Newton was only followed in the British Isles where the development of mathematics was clearly slower than on the Continent, evidently at least partly for the lesser suggestivity of Newton's notations. – Another reason for the slowing down of mathematical development in England was the unlucky priority

dispute which was going on from 1699 onwards. The main accusation was that Leibniz had copied Newton's ideas. The English mathematicians refused, for patriotic reasons, to use the Leibnizian notation or results of Leibnizian analysis. In fact, Newton and Leibniz had in 1676 and 1677 a brief exchange of letters on the foundations of the differential calculus, which both men knew at that time. There is no reason to believe that either had copied from the other.

## 9 The 18th century: rapid development of analysis

The invention of the infinitesimal calculus gave a strong impulse for rapid development of mathematical analysis. The new computational, "algebraic" methods were sometimes used uncritically and little attention was paid to their logical foundations. From a purely technical point of view, differential and integral calculus in many ways reached its modern level. On the other hand, in the 18th century mathematics did not experience changes as revolutionary as the ones of the previous century.

### 9.1 The Bernoullis

The Bernoulli family originally came from the Netherlands, but settled in Basel at the end of the 16th century. The family is probably the most important single family in the history of mathematics and science. It contains about a dozen first class scientists. Of the mathematicians in the family, the most famous are the brothers *Jakob* (1654–1705) and *Johann* (1667–1748) *Bernoulli*. (Their first names may also appear as e.g. *Jacques* and *Jean* or *James* and *John* in different languages.) The Bernoulli brothers were students, collaborators and also rivals of Leibniz. It was very much due to their influence that the differential and integral calculus was propagated in the format created by Leibniz.

Jakob Bernoulli presented the *Bernoulli inequality*  $1 + nx < (1 + x)^n$ , proved that the harmonic series diverges (the ear-

lier proof by Oresme was forgotten) and investigated various curves using the calculus. The word *integral* was introduced in 1690 by Jakob Bernoulli. Leibniz had used the expression *calculus summatorius*, but he subsequently took up the word integral. The Bernoullis invented many of the usual integration techniques. To calculate, for instance, the integral

$$\int \frac{a^2 dx}{a^2 - x^2},$$

Jakob Bernoulli observed that the substitution

$$x = a \frac{b^2 - t^2}{b^2 + t^2}$$

works, while Johann discovered the more effective partial fraction decomposition

$$\frac{a^2}{a^2 - x^2} = \frac{a}{2} \left( \frac{1}{a + x} + \frac{1}{a - x} \right).$$

One of the problems Leibniz and the Bernoullis were much interested in was the *brachistocron problem*. The purpose was to find a curve along which a body influenced by gravity would slide in the shortest possible time from point  $A$  to point  $B$ , not directly below  $A$ . It was Jakob Bernoulli who was the first to show that the solution curve is an arc of a cycloid. The brachistocron problem started the *calculus of variations*, the branch of mathematics which seeks functions and not only values of a function as solutions to extremal problems. In fact, the first discovery in this field was made by Newton: his solution of the problem concerning the shape of a body with least resistance in a medium preceded Jakob Bernoulli's work.

Other early variational problems were the *isochron problem*, the problem of finding a curve along which a body moving

under gravity uses the same time to reach a given point not depending on the initial point, and the *catenary problem* or the question of the form assumed by a hanging rope or chain. The catenary problem was first introduced by Leonardo da Vinci, and Galileo Galilei had claimed that the solution is a parabola. Johann Bernoulli derived the differential equation of the catenary, but he could not integrate it. – The solution has turned out to be the graph of a hyperbolic cosine.

Jakob Bernoulli also pondered the series

$$\sum \frac{1}{n^2}.$$

He knew that it was convergent because of the majorant series

$$\sum \frac{1}{n(n-1)}.$$

The Bernoullis and their contemporaries often treated infinite series in a rather casual manner. Although Jakob Bernoulli had proved that the harmonic series diverges, he concluded from the "identities"

$$N = \frac{a}{c} + \frac{a}{2c} + \frac{a}{3c} + \dots, \quad N - \frac{a}{c} = \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \dots$$

that

$$\frac{a}{1 \cdot 2c} + \frac{a}{2 \cdot 3c} + \frac{a}{3 \cdot 4c} + \dots = \frac{a}{c}.$$

The conclusion is correct.

The long list of Jakob Bernoulli's accomplishments includes the introduction of polar coordinates and the presentation and solution of the so called *Bernoulli differential equation*  $y' + p(x)y = q(x)y^n$ . The equation was also solved at the same time by Leibniz and brother Johann. Jakob Bernoulli also wrote the first real monograph on probability, *Ars conjectandi*. It appeared posthumously in 1713. (Huygens had

published in 1657 a booklet on probability, *De ratiociniis in ludo aleae*.) Bernoulli's book contains a systematic treatment of the elements of combinatorics, an induction proof of the binomial formula, the definition of the *Bernoulli numbers*  $B_k$ , (they show up e.g. in the expressions for sums of even powers of consecutive integers) and the *law of the large numbers* in probability.

Johann Bernoulli was often a rival of his elder brother and they sometimes were in deep controversies with each other. Johann eventually succeeded his brother as professor in Basel. Before that, he collaborated for some time with the French Marquis *Guillaume l'Hôpital* (1661–1704). The Marquis employed Johann Bernoulli, one of the less than half dozen people at that time well versed in the new differential and integral calculus, as a teacher. Part of the agreement they had gave the Marquis the right to assume to his name all new discoveries by Johann. One of these was the observation, known as *l'Hôpital's rule*, that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)},$$

if  $f(a) = g(a) = 0$ . In fact, l'Hôpital and Johann Bernoulli do not speak of limits but of the value of an expression of the type  $\frac{0}{0}$ . The Marquis published the rule and other things that Johann Bernoulli taught him in 1696 in the book *Analyse des infiniment petits*. It was the first textbook in differential calculus. The book was very influential in the 18th century, although many of its premises (like that roots of an equation differing only by an infinitely small quantity or that curves are made from infinitely short line segments) are not quite convincing at present time. The production of Johann Bernoulli published under his own name is large: it includes works on the calculus of variations, differential

geometry, the first observation relating trigonometric and exponential functions and first instances of functional notation reminiscent of the modern one. (Bernoulli denoted a function of  $x$  by  $\phi x$ ).

Johann Bernoulli was one of the most aggressive proponents of Leibniz in the differential and integral calculus priority dispute. His son *Daniel Bernoulli* (1700–82) was a versatile mathematician and scientist. As mathematician he is primarily remembered as one of the initiators of the study of partial differential equations. *Nicolaus Bernoulli* (1687–1759), a nephew of Jakob and Johann, was the first to treat the total differential  $df = p dx + q dy$  of a function of two variables.

## 9.2 18th century mathematics in Britain

Newton and his legacy had a dominating but restraining influence on the mathematics in England in the 18th century. However, the century is by no means a mathematical vacuum in the British Isles. The most important mathematicians, active in Britain were *Abraham de Moivre* (1667–1754), a French immigrant who had left his country for religious reasons and spent most of his life in poverty, and *Colin Maclaurin* (1698–1746), a Scot and a pupil of Newton.

De Moivre is one of the pioneers of probability theory. The error function  $e^{-x^2}$  is first seen in his production and his book *Doctrine of Chances* (1718) is together with Jakob Bernoulli's *Ars conjectandi* the first systematic treatment of the calculus of probability. De Moivre was more familiar with complex numbers than his predecessors. The formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

the *de Moivre formula*, was not written explicitly so by de Moivre, but in an almost equivalent form. On the other

hand, the approximation

$$n! \approx \sqrt{2\pi n} n^n e^{-n},$$

which got its name *Stirling's formula* from *James Stirling* (1692–1770), a Scottish contemporary of de Moivre and completer of Newton's studies on cubic curves, was actually discovered by de Moivre and used by him in the study of the binomial distribution. De Moivre is one of the pioneers of insurance mathematics.

Maclaurin's name is associated with the *Maclaurin series*

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots$$

In fact, the series had been published by the Englishman *Brook Taylor* (1685–1731) in its more general form, that of a *Taylor series* already in 1715, and *James Gregory* (Scottish, 1638–75) had used it in special cases. The series was also known to Newton and Johann Bernoulli. – In the formulations of Taylor, Maclaurin and Newton, the derivatives of  $f$  are replaced by corresponding quotients of fluxions.

The real merits of Maclaurin lie in the study of curves of higher degree, (which was initiated by Newton) and in writing of the first textbooks dealing with Newtonian calculus. Maclaurin's book, *Treatise of Fluxions*, appeared in 1742. The book tried to present the Calculus with the "rigour of the ancients". Maclaurin defended Newton in the polemics concerning the tenability of the fundamentals of the calculus. His opponent was a well-known philosopher, the Irish bishop *George Berkeley* (1685–1753). Berkeley questioned – with good reason – the infinitely small magnitudes which were considered zeros or non-zeros whatever was more convenient (he called them mockingly "*ghosts of departed quantities*"), and assumed that the correct results obtained by using the

calculus resulted from coincidences and errors neutralizing each other. Berkley was as critical towards the Leibnizian analysis. According to Berkley, the truths of mathematics were not standing on a firmer ground than those of religion. – Maclaurin also wrote a popular textbook of algebra. His *Treatise of Algebra* appeared only posthumously in 1748 but it reached six printings. The solution formula for systems of two or three linear equations, known as *Cramer's rule* appeared in Maclaurin's book for the first time. The name comes from *Gabriel Cramer* (1704–52), a Swiss mathematician who used indexed coefficients in the equations and thus made the symmetry of the rule more apparent in his treatise on the matter. It appeared in 1750.

According to a general opinion, Maclaurin's strict adherence to classical methods slowed down the development of mathematics in England. – Maclaurin caught a fatal illness when he participated in the defence of Edinburgh in 1745 on the side of the English against revolting Scots.

Another English textbook writer has left his name in the mathematical vocabulary. He was *Thomas Simpson* (1710–61), professor in the Royal Woolwich Military Academy. His book *A New Treatise of Fluxions* (1737) was ordered by his private pupils. However, the *Simpson rule* of approximate integration had been previously discovered at least by Stirling.

### 9.3 Euler

One of the most important mathematicians of all times is *Leonhard Euler* (1707–83). He shared with the Bernoullis the home town, Basel, and he was a student of Johann Bernoulli. Euler, like several other important mathematicians of the 18th century, did his life's work in scientific academies established by sovereigns and closely connected

to the courts. Euler's longest standing employer the St. Petersburg Academy which was established by the Czar Peter the Great. He was persuaded to go to St. Petersburg by Johann Bernoulli's sons Daniel and *Nicolaus Bernoulli* (1695–1726) who had established themselves there earlier. Euler spent also several years in Berlin in the Prussian Academy, founded by Frederick the Great, but when Catherine the Great asked him back, he returned to St. Petersburg for his final years. His burial place is in the cemetery of the monastery of Alexander Nevskiy.

Euler's productivity is difficult to comprehend. Although he lost one eye when he was under 30 and became totally blind 17 years before his death, he wrote or dictated mathematics without interruption, 800 pages a year in the average. In his lifetime Euler published over 500 studies and they continued to appear for 40 years after his death. The total number of his publications has been counted as 856. His first scientific paper appeared when he was 19. It dealt with a question concerning the masts of a sailboat – a question which does not seem very relevant in Switzerland. The publication of Euler's Collected Works is not yet completed. The collection is projected to consist of 72 thick volumes. Thousands of letters, unpublished manuscripts and diaries are not included. There are numerous concepts in mathematics and physics named after Euler. – Euler had 13 children.

Euler's publications deal with almost every branch of the mathematics and physics of his time, and besides scientific articles they include textbooks and popular expositions. Euler's *Letters to a German Princess* (1760–61) is an early model of popularization of science. Euler created a large part of the established notations of mathematics. From him we have the letter  $e$  as the base of the natural logarithms (Euler proved the irrationality of  $e$  and was the first to consider logarithms as exponents),  $\pi$  as the ratio of the circum-

ference and diameter of a circle,  $i = \sqrt{-1}$  as the imaginary unit, the functional notation  $f(x)$  (from 1734), the standard notation of the sides and angles of a triangle as  $a, b, c$ ;  $A, B, C$ , the sum sign  $\sum$ , the binomial coefficient notation  $\binom{p}{q}$

(to be exact, Euler wrote here  $\left[\frac{p}{q}\right]$ ), the standard notations  $R$  and  $r$  for the radii of the circumscribed and inscribed circles of a triangle. The  $\pi$  symbol, however, had been used earlier by a certain Englishman *William Jones* (1675–1749).

Not restricting to only notations, we may say that Euler's numerous textbooks, like *Introductio in analysin infinitorum* (1748), *Institutiones calculi differentialis* (1755), *Institutiones calculi integralis* (1768–74) and the German *Vollständige Anleitung zur Algebra* (1770), created much of the established canon of teaching university level mathematics. The *Introductio* assembled the fundamentals of the new analysis around the concept of a function. This concept did not reach complete clarity in Euler's exposition. He defined it sometimes as an expression arbitrarily put together from variables and constants, sometimes as a dependency concretized by an arbitrary curve, "drawn with a free hand", in the coordinate plane. – Euler's arguments were not always correct: from the sum formula of the geometric series he drew the straightforward conclusion

$$\dots x^{-2} + x^{-1} + 1 + x + x^2 + \dots = \frac{\frac{1}{x}}{1 - \frac{1}{x}} + \frac{1}{1 - x} = 0!$$

In most cases, Euler's intuition guided him to a correct conclusion even if some steps in the argument were deficient.

In any case, Euler's most important tools were infinite series.

For instance, the equation

$$\frac{\sin \sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \dots = 0$$

has  $x_1 = \pi^2$ ,  $x_2 = (2\pi)^2$ , ... as solutions. But it is well known that the roots  $x_1, x_2, \dots, x_n$  of the algebraic equation  $1 + a_1x + \dots + a_nx^n = 0$  satisfy

$$1 + a_1x + a_2x^2 + \dots + a_nx^n = \left(1 - \frac{x}{x_1}\right) \left(1 - \frac{x}{x_2}\right) \dots \left(1 - \frac{x}{x_n}\right)$$

and

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = -a_1.$$

Extending this result to "polynomials of infinite degree" one would obtain

$$\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \dots = -\left(-\frac{1}{3!}\right),$$

whence

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

When Johann Bernoulli was informed of Euler's result he said that he very much wished that his brother would have been alive. Jakob Bernoulli had devoted much trouble in futile attempts to calculate the sum of the inverses of squares. With similar arguments Euler derived the sum of the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

later known as the Riemann zeta function, for even integral  $s$ . For  $s = 1$ ,  $\zeta(s)$  is the divergent harmonic series. Euler proved that the difference of the  $n$ :th partial sum of the

harmonic series and  $\ln n$  approaches the limit  $\gamma \approx 0,577218$  when  $n$  tends to infinity. The number  $\gamma$  is known as the *Euler constant*. Euler discovered the important number theoretic identity

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the product is over all prime number  $p$ .

As another example of Euler's arguments we outline his derivation of the basic properties of the exponential function. If  $\epsilon$  is an "infinitely small number", then  $a^\epsilon = 1 + k\epsilon$ , where  $k$  is a constant depending on  $a$ . If now  $x$  is a finite number, then  $N = \frac{x}{\epsilon}$  is an "infinitely large number". Then

$$\begin{aligned} a^x &= a^{N\epsilon} = (1 + k\epsilon)^N = \left(1 + \frac{kx}{N}\right)^N = 1 + N \left(\frac{kx}{N}\right) \\ &+ \frac{N(N-1)}{2!} \left(\frac{kx}{N}\right)^2 + \frac{N(N-1)(N-2)}{3!} \left(\frac{kx}{N}\right)^3 + \dots \\ &= 1 + kx + \frac{1}{2!} \frac{N(N-1)}{N^2} k^2 x^2 + \frac{1}{3!} \frac{N(N-1)(N-2)}{N^3} k^3 x^3 + \dots \end{aligned}$$

But since  $N$  is infinitely large,

$$\frac{N-1}{N} = \frac{N-2}{N} = \dots = 1,$$

and

$$a^x = 1 + \frac{kx}{1!} + \frac{k^2 x^2}{2!} + \frac{k^3 x^3}{3!} + \dots$$

With  $x = 1$  one obtains

$$a = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \dots$$

Now denote by  $e$  the  $a$  which corresponds to  $k = 1$ . Then

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

By what was written earlier,  $e^x = \left(1 + \frac{x}{N}\right)^N$ .

Euler operated with complex numbers without scruples. The *Euler formula*

$$e^{ix} = \cos x + i \sin x$$

(which can be ascribed to de Moivre) was a central tool for him e.g. when dealing with the properties of the logarithmic function: Euler cleared the long-standing problem of the logarithms of negative numbers and realized the one to many property of the logarithm function. Euler was one of the pioneers of calculus in several variables. The formula for transformation of variables in a double integral as well as the reduction of a double integral to successive one-dimensional integrations were presented by him.

On a more advanced level, Euler initiated the study of *elliptic integrals*, which was to become a significant branch of analysis somewhat later. Problems leading to elliptic integrals, i.e. integrals of functions containing square roots of polynomials of degree three or higher, had been encountered already before 1700 by the Bernoullis among others. Such problems include the shape of an elastic rod, the movement of a pendulum, the arc length of a lemniscate,  $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ , and above all the arc length of an ellipse, a problem of major interest in astronomy. The latter had been investigated by Count *Carlo de'Toschi Fagnano* (1682–1766) in Italy, but the first significant addition theorem of elliptic integrals,

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} = \int_0^y \frac{dt}{\sqrt{1-t^4}} + \int_0^c \frac{dt}{\sqrt{1-t^4}},$$

if

$$x = \frac{y\sqrt{1-c^4} \pm c\sqrt{1-y^4}}{1+c^2y^2},$$

was discovered by Euler.

Euler's influence can be seen in the theory of ordinary differential equations and calculus of variations. The established order of presentation of the subject matter in courses of differential equations as well as many techniques like using of integrating factors and the solution formulas of linear equations with constant coefficients come from him. In the calculus of variations, the necessary maximality condition for the a function  $y = y(x)$  to maximize the integral

$$I(y) = \int_a^b F(x, y, y') dx$$

is the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0,$$

again *Euler's equation*.

Euler's influence on other parts of mathematics than analysis is also great. Euler developed number theory to a great extent. He showed that Fermat's claim that all integers of the form  $2^{2^n} + 1$ , or Fermat's numbers are prime, was incorrect. He gave the first proof of the so called Fermat's Small Theorem, which he also generalized using the number theoretic function now known as the *Euler function*  $\phi$  (the name was given by Gauss). ( $\phi(n)$  is the number of positive integers less than  $n$  which are relatively prime to  $n$ .) He proved that the series whose terms are inverses of primes diverges (thus giving another proof for the infinity of primes) and he proved Fermat's Last Theorem for the exponent  $n = 3$ , showed that all perfect numbers are of the form

given by Euclid etc. In geometry Euler left the *Euler line*, the line joining the orthocentre, centre of gravity and circumcentre of a triangle, and the *Euler formula*  $v - e + s = 2$  for the numbers  $v$  of vertices,  $e$  of edges and  $s$  of sides of an arbitrary simply connected polyhedron. Euler also has to his credit the determination of the properties of surfaces of the second degree, the three-dimensional analogy of conic sections.

One of Euler's most famous achievements is his solution of the *Königsberg bridge problem* (1736). The problem calls for a route which crosses each of the seven bridges over the river Pregel in Königsberg (now Kaliningrad), each one only once. Euler showed that the problem has no solutions. Although the problem and its solution can be classified as recreational mathematics, Euler's work has in a sense marked the birth of two later important branches of mathematics, topology and graph theory. In graph theory, an *Euler path* is a traverse of a graph in which every edge of the graph is traversed exactly once.

#### 9.4 Enlightenment mathematicians in Italy and France

The Italian *Maria Gaetana Agnesi* (1718–99) taught mathematics to her younger brothers. She published her material in 1748 as *Instituzioni analitiche ad uso della giovena italiana*. Agnesi's name remains in the history of mathematics not only because of this good textbook but also because of a misunderstanding. Agnesi considered as an example the curve

$$y = \frac{a\sqrt{a-x}}{\sqrt{x}}$$

which was known already before her as *la versiera*, 'turning'. When the book was translated into English, the translator

mistook the word for *avversiera*, 'devil's wife'. From then on, the curve has been known as the *Witch of Agnesi*.

The most important French mathematicians active around the middle of the 18th century were *Alexis Claude Clairaut* (1713–65) and *Jean le Rond d'Alembert* (1717–83). The former – one of the most precocious mathematicians ever – read l'Hôpital when he was ten and published a treatise on three-dimensional analytic geometry in his teens. He was chosen, by exemption, to the French Academy of Sciences at the age of 18. – Clairaut even visited Finland as a member of the Maupertuis expedition to Lapland. The purpose was to measure the length of the meridian and infer the shape of the Earth. Clairaut later published a treatise on this matter. The work also contains results in the theory of differential equations, among others the exactness condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for the differential  $Mdx + Ndy$ .

D'Alembert was a foundling. His first name Jean le Rond comes from the Paris church on the stairs of which his well-to-do mother had abandoned the son. The father, an artillery officer, took care of the son's education. D'Alembert was a versatile scientist and philosopher, originally a jurist. After passing the examination in law, he wanted to study medicine. He considered his interest in mathematics to be detrimental to his studies, so he left his mathematics books to a friend. But he could not resist mathematics, but eventually abandoned law and medicine in favour of mathematics and philosophy. His longest lasting employment was that of the influential permanent secretary of the French Academy of Sciences. It was d'Alembert who advised Frederick to invite Euler to Berlin as well as later Lagrange.

D'Alembert played a central role in the publication of the magnum opus of the enlightenment philosophy, the 28 volume *Encyclopédie* whose chief editor was *Denis Diderot* (1713–84). D'Alembert was responsible for the articles on mathematics and natural sciences. It was in an encyclopaedia article that d'Alembert expressed his modern opinion according to which the infinitesimal calculus should be founded on an exact limit concept. D'Alembert defined the limit to be a quantity towards which a variable quantity approaches so that the difference of the quantity and the limit becomes smaller than any given quantity. D'Alembert tried to find a proof for the *Fundamental Theorem of Algebra* (which states that every non-constant polynomial has at least one complex zero). The Francophone name for this result is *d'Alembert's Theorem*. Gauss succeeded in giving a more rigorous proof of the theorem.

D'Alembert is along with Euler and Daniel Bernoulli one of the initiators of the theory of *partial differential equations*. He studied the behaviour of a vibrating string (a problem which has given rise to much mathematical theory and created many disputes among the leading mathematicians of the 18th century. D'Alembert came across the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

For this *wave equation* d'Alembert found the solution  $u(x, t) = f(x + t) + g(x - t)$ ; here  $f$  and  $g$  are arbitrary functions.

*Etienne Bezout* (1730–83) made significant advances in the theory of determinants initiated by Leibniz. One of his observations was that the vanishing of the determinant of a homogeneous system of linear equations is necessary for the existence of non-trivial solutions. Bezout also developed the theory of resultants or conditions under which two polyno-

mial equations have a common root. Bezout's method enabled one to calculate the number of intersections of two algebraic curves.

## 9.5 Lagrange

The most significant French mathematician of the latter half of the 18th century was *Joseph Louis Lagrange* (1736–1813), a rival to Euler for the title of the greatest mathematician of the era. Lagrange was half Italian. He was born in Turin and studied there. Already at 19 he was appointed professor of mathematics in the Artillery Academy in Turin. Lagrange's most important employers were sovereigns, as the case was with Euler, too. Frederick the Great invited Lagrange to succeed Euler in the Berlin Academy. Lagrange spent 20 years in Berlin but in 1787 he was invited to Paris by Louis XVI. The rest of his life he spent in Paris. Lagrange suffered from depression, especially after moving to France.

Lagrange was more critical in his research than most other mathematicians of his time. He tried to fill the gaps in the foundations of analysis in his great *Théorie des fonctions analytiques* (1797) by a systematic use of power series. If the Taylor series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

of  $f$  is known, then the derivatives of  $f$  can be expressed in terms of the coefficients:  $f'(x_0) = a_1$ ,  $f''(x_0) = 2a_2$  etc. By making these relations the definitions of derivatives, Lagrange believed to have avoided the infinitesimal quantities. The word *derivative* was first used by Lagrange. (This choice of word is somewhat unlucky at least from the pedagogical perspective, since it fails to exhibit the basic meaning of the derivative as the rate of change.) Unfortunately,

Lagrange's definition only works with functions with a representation as a convergent power series. The well-known counterexample  $f(x) = e^{-1/x^2}$ ,  $x \neq 0$ ,  $f(0) = 0$  shows that a function non-zero for  $x \neq 0$  can have all its derivatives zero at the origin, in which case the Taylor series is that of the zero function. Lagrange did not take into account the questions of convergence which inevitably are linked with limits. – The usual derivative denotations  $f'(x)$ ,  $f''(x)$  etc. were introduced by Lagrange as well as the first expression for the rest term of the Taylor series:

$$f(x) = f(x_0) + \sum_{k=1}^n f^{(k)}(x_0)(x - x_0)^k + R_n,$$

$$R_n = f^{(n+1)}(\xi) \frac{(x - x_0)^{n+1}}{(n + 1)!},$$

where  $\xi$  lies between  $x$  and  $x_0$ . Also the coordinate transformation  $dx dy dz = r^2 \sin \theta dr d\theta d\phi$  for integration in spherical coordinates comes from Lagrange.

A major study area of Lagrange was mechanics as we are reminded by the *Lagrangian* and the Lagrange equations of motion. His monumental *Mécanique analytique* (1788), is an axiomatic presentation of mechanics "without a single illustration". Related to this were Lagrange's results in the theory of differential equations (the method of solving nonhomogeneous linear equations by varying the parameters is his discovery) and the calculus of variations. (The name calculus of variations comes from the word method of variations used by Lagrange in his early publications; the present name was then formulated by Euler). The solution of an extremum problem in the presence of side conditions to the function to be optimized by the method of *Lagrange multipliers* is one of well-known Lagrange results.

Lagrange also worked in number theory – he proved the Fermat conjecture which says that every positive integer is the sum of at most four squares, gave a general method for solving Pell’s equations and gave the general solution for quadratic Diophantine equations. In algebra Lagrange is a predecessor of the group concept. He studied the connections between the solvability of a polynomial equation and permutation properties of its roots. In a sense he proved the central theorem in group theory which states that the order of a subgroup of a finite group is a factor of the order of the full group. We should note of course that the group concept as such was not known at Lagrange’s time. Lagrange concluded, that a polynomial equation of degree higher than four most likely has not an algebraic solution.

After the French Revolution Lagrange worked as chairman of the committee which planned the metric system. Lagrange pushed through the ratio 10, although 12 also had much support.

## 10 Mathematics at the time of the French Revolution

The turn of the 18th and 19th centuries was a mathematically active period, especially in France. Lagrange was still productive at that time, and several mathematicians treated here started their career already before the Revolution. Gauss, often appreciated as the greatest mathematician of all times, started his work at the time of the French revolution although he was not much influenced by the political turmoil of the time. The central political and military figure of the time, *Napoleon Bonaparte* (1769–1821) is probably the only ruler who left his name in the mathematical vocabulary: *the Theorem of Napoleon* states that if equilateral triangles are constructed on the sides of an arbitrary triangle, then the centres of gravity of these triangles are again vertices of an equilateral triangle. A quote from an old encyclopaedia entry tells us that "At school he showed a great inclination towards mathematics, and he was interested in history and geography, but he was weak in languages."

### 10.1 Monge and École Polytechnique

*Gaspard Monge* (1746–1818) was a largely self-educated mathematician. He is known as a developer analytic and differential geometry and as the inventor of *descriptive geometry*. This field of applied mathematics has been important in the development of technology. Monge discovered the method as a young drawer in a military institution – de-

scriptive geometry replaced tedious numerical computations needed in the construction of fortresses and it was kept as a military secret by the Royal Mézières Engineer Academy and the French Army. It was not before 1800 that Monge was able to publish his method. Descriptive geometry is a synthetic method. Monge was also important in analytic geometry. The equations of a line in three-dimensional space come essentially from him. He is also regarded as one of the founders of differential geometry.

Monge took an active part in the French Revolution. He was given important government positions – he served as Minister of Navy, among other things. In the period of the Directorate, in 1794, Monge was instrumental in the creation of an institute of higher technical studies, the *École Polytechnique* in Paris. Monge later served as director and main geometry teacher of the *École Polytechnique*. *École Polytechnique* is the oldest technical institution of its kind. The growing importance of technology instigated the founding of similar institutes also elsewhere. A technological university was founded in Berlin in 1799, in Prague at 1806 etc. The *École Polytechnique* was very influential in the development of mathematics and higher mathematical instruction. The level of the courses was high, both what comes to mathematical content and requirements set to the students. The *École Polytechnique* became the uncontested centre of French mathematics. Its early faculty included along with Monge also Lagrange, Legendre and Laplace. The significance of mathematics for technology started to change the position of mathematics in society – mathematics on the way of becoming a practically useful science.

Soon another important institution, the *École Normale* was founded to take care of the education of future teachers. This second *Grand École* also counted important mathematicians in its faculty. Textbooks written for the mathe-

matics courses of the *Écoles* were unquestionably the top of their kind in the 19th century. The ancient Sorbonne, the University of Paris, was of minor importance for mathematics in the 19th century.

## 10.2 Fourier

An important friend and collaborator of Monge as well in the *École polytechnique* as on Napoleon's Egyptian expedition was *Joseph Fourier* (1768–1830) (Monge was the director of the Egyptian Institute founded by Napoleon and Fourier was its secretary; Fourier also oversaw the publication of the scientific material collected during the expedition.) Fourier's plan was to enter a military career, but he had to abandon his hope of becoming an officer because of his low birth. Instead, he was educated for priesthood. He then got a position better suited to his abilities, when he was appointed to teach mathematics at a military college.

Fourier made a revolutionary observation which was also severely opposed: every function  $f$ , not necessarily even continuous, has a representation as a trigonometric series

$$\sum_{n=0}^{\infty} (a_n \cos(nax) + b_n \sin(nax)).$$

The series representation has later got the name *Fourier series* of  $f$ . The main work of Fourier, *Théorie Analytique de la Chaleur* (Analytic theory of heat, 1822), uses trigonometric series in the solution of boundary value problems of partial differential equations. Fourier's arguments were not logically flawless, but efforts to amend them were important motives for the 19th century program to set the foundations of analysis on a firm basis.

The starting point of Fourier's series development is the search for a stable distribution of heat in the domain

$\{(x, y) \mid 0 < x < \pi, 0 < y\}$ . Because of physical reasons, the distribution  $u(x, y)$  satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the boundary conditions  $u(0, y) = u(\pi, y) = 0$ ,  $u(x, 0) = \phi(x)$ , where  $\phi$  is a given function. If the solution is of the form  $u(x, y) = f(x)g(y)$ , particular solutions of the form  $u(x, y) = e^{-ny} \sin(nx)$ ,  $n = 1, 2, 3, \dots$  are obtained and a more general superpositional solution is  $u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin(nx)$ , where the  $b_n$  are constants. If the constants can be determined to satisfy

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \phi(x),$$

the solution is there. An argument mostly based on the Taylor series of  $\phi$  led Fourier to think that the coefficients satisfy a second order differential equation which finally leads to the expression

$$b_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin(nx) dx.$$

At this point Fourier realized that he will get the same result by assuming the trigonometric development of  $\phi$  and using the "orthogonality conditions"

$$\int_0^{\pi} \sin(kx) \sin(nx) dx = 0, \text{ if } k \neq n.$$

The expressions only presuppose information of "the area under a curve", and periodicity easily introduces points of discontinuity. Thus Fourier's series innovation motivated investigation of integration and other processes of analysis

also in the realm of discontinuous functions. The convergence of Fourier's series and the behaviour of the sum function are not straightforward. These questions had a great influence in the development of analysis in the 19th century. – Fourier's work also contains first hints towards the important *Fourier integral* or *Fourier transformation* of a function  $f$ .

On the level of mathematical notation, Fourier introduced the way a definite integral is denoted:  $\int_a^b f(x) dx$ . Euler's notation had been  $\int f(x) dx \left[ \begin{array}{l} x = a \\ x = b \end{array} \right]$ .

### 10.3 Laplace and Legendre

*Pierre Simon Laplace* (1749–1827), "The Newton of France", commenced his mathematical career as teacher of the military academy *École Militaire* in Paris. The post was obtained to him by d'Alembert. d'Alembert's attitude towards Laplace was originally negative, but Laplace succeeded in convincing d'Alembert of his abilities by submitting a comprehensive treatise of the foundations of mechanics. Laplace, too, was socially active at the times of the revolution and Napoleon, and he was Napoleon's minister of the interior for some time. Laplace's governmental career was not quite successful. Napoleon dismissed the over-pedantic Laplace, because he had "brought the spirit of the infinitesimals into government". Laplace adapted himself flexibly to the rapid political changes.

Laplace wrote two great treatises: *Théorie analytique des probabilités* (1812) and *Mécanique céleste* (five volumes, 1799 – 1825). The former is a complete presentation of probability using all tools from mathematical analysis. It contains several remarkable innovations such as the *Laplace*

*transformation*

$$\mathcal{L}f(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

This tool transforms many differential equation to algebraic equations. Another is the *method of least squares* to smooth errors in observations. The method had been published by Legendre, without proof. The preface of the second edition of Laplace's work contains his thesis according to which future is determined by the past. If we could have sufficient information, we could compute the state of the world at any given moment of the future.

Laplace's monumental presentation of celestial mechanics gives a general view of the physics of the solar system according to Newton's principles. In mathematics, the work introduces the *potential function* (which was known to Lagrange, too), the *Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and the *Laplace differential equation*  $\Delta u = 0$ , whose solutions are potential functions. To Laplace, mathematics was always of secondary importance: it was indispensable for science, in particular for celestial mechanics, but it had little value outside its applications.

*Adrien Marie Legendre* (1752–1833) also started as teacher in the *École Militaire*. He maintained a distance from the tasks of the society except his contribution in the metric system project which also employed Lagrange, Monge and Laplace. The units in the decimal system were completed in 1799.

Legendre wrote good and popular textbooks in geometry and calculus, among others. Mathematical physics is grateful to him for several mathematical tools such as the useful *Legendre functions*, solutions of the differential equation  $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ . Legendre developed and systematized the theory of *elliptic integrals* or integrals of the type

$$\int R(x, \sqrt{s(x)}) dx,$$

where  $R$  is a rational function and  $s$  is a polynomial of degree 3 or 4. He showed that these integrals can always be reduced to a few of canonical forms, *Legendre normal forms*. The most important of these are

$$F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

and

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 x} dx.$$

The values of  $E$  and  $F$  have been tabulated. The elliptic integrals were an important object of study in the 19th century; their significance for the emergence of the general theory of functions of a complex variable has been great.

Legendre's work in the theory of numbers was also important. His works on quadratic residues are well-known. They gave rise to the *Legendre symbol*  $(p|q)$ , which equals  $+1$ , if  $p$  is the residue of a square number modulo  $q$  and  $-1$  in the opposite case. Based on empirical observations, Legendre formulated the *prime number hypothesis*, according to which  $\pi(n)$ , the number of primes smaller than  $n$ , approaches asymptotically the value

$$\frac{n}{\ln n},$$

as  $n$  increases. The hypothesis is unexpected as the distribution of primes seems to be rather accidental, but it was proved in 1896 by *Jacques Hadamard* (French, 1865–1963) and *Charles de la Vallée Poussin* (Belgian, 1866–1962); in the mean time, the Russian *Pafnuti Chebyšev* (1821–94) had proved that the limit

$$\lim_{n \rightarrow \infty} \pi(n) \frac{\ln n}{n}$$

exists. – Legendre gave a proof to Fermat’s last theorem in the case  $n = 5$ .

## 10.4 Gauss

*Carl Friedrich Gauss* (1777–1855), ”*Princeps Mathematicorum*”, was born to a working class family in Braunschweig near Hannover (exactly 50 years after the death of Newton, as somebody has noticed). Although his parents opposed, the precocious youth acquired schooling, supported by the local Duke. At the age of 18, Gauss hesitated between careers as a philologist or mathematician, although he had already discovered the method of least squares, 10 years before Legendre. His choice was clear after he found out how to construct a regular 17-gon with straightedge and compasses. No construction of a  $p$ -gon for odd  $p$  other than 3 and 5 had been done in the two millennia of plane geometry. (Gauss wished to have a 17-gon in his tombstone, in the vein of Archimedes, sphere, cylinder and cone. The Gauss memorial in Braunschweig, however, displays a 17-pointed star, because the artist thought that a 17-gon looks too much like a circle.)

Like Newton, Gauss also was reluctant to publish his results. Even his seal has the motto *Pauca sed matura*, ‘little, but mature’. Investigation of Gauss’s survived notes has

revealed that he made independently several significant inventions ascribed to other mathematicians.

After the note on the 17-gon, Gauss's next publication was his thesis in 1799. It contained an acceptable proof of the *fundamental theorem of algebra* (the name is due to Gauss). (A really exact proof of course requires an exact definition of a real number, which did not yet exist at Gauss's time.) Earlier attempts to prove the theorem had been made, beside by d'Alembert, also by Euler and Lagrange. Gauss later returned to the question and presented three additional different proofs. Gauss's success in the proof of the theorem is connected with his observation to represent complex numbers as vectors in the plane. (The same observation was made independently at the same time by the Norwegian surveyor *Caspar Wessel* (1745–1818) and the Swiss accountant *Jean Robert Argand* (1768–1822); the graphical representation of complex numbers is sometimes called the *Argand diagram*.) – Gauss gave a definition of complex numbers also using congruences of polynomials  $\text{mod } x^2 + 1$ .

The proof by Gauss is based on the consideration of the equation  $Q(x, y) + iR(x, y) = 0$ , equivalent to the original equation  $P(x + iy) = 0$ . The first equation is satisfied at the intersection of the curves  $Q(x, y) = 0$  and  $R(x, y) = 0$ ; Gauss proved that these curves always lie in such a position that they have to intersect. The proof is one of the first instances of a pure existence proof: there is no formula or algorithm to get the value of the root. Of the later proofs, the last one was most general as it allowed the coefficients of the polynomial to be complex numbers, too.

Gauss's most famous mathematical work is the number-theoretic treatise *Disquisitiones arithmeticae* which appeared two years after the thesis, in 1801. The book introduces the *number congruence* and the notation  $a \equiv b$

(mod  $p$ ), proves the *quadratic reciprocity theorem*

$$(p|q)(q|p) = (-1)^{(p-1)(q-1)/4}$$

which was also discovered by Legendre and anticipated by Euler (Gauss gave six different proofs of the theorem in his lifetime) and gives a necessary and sufficient condition to the constructability of a regular  $n$ -gon with Euclidean tools ( $n$  has to be the product of a power of 2 and Fermat primes  $2^{2^{n_k}} + 1$ ). (A certain *Oswald Hermes* (1826–1909) used 10 years of his life to realize the construction of a regular 65537-gon.)

When generalizing the quadratic reciprocity theorem for residues of higher powers, Gauss was induced to consider divisibility of complex integers or *Gaussian integers*  $a + ib$ , where  $a$  and  $b$  are ordinary integers. Their divisibility properties are different from those of ordinary integers. For instance  $5 = (2 + i)(2 - i)$  is no more a prime among Gaussian integers. Most likely the *prime number hypothesis* was known to Gauss, but he never published anything of it.

Gauss's longest lasting position was that of the Director of the Observatory of Göttingen. At the same time, he was professor of mathematics in the Göttingen University. Gauss stayed in Göttingen for the rest of his life, despite invitations from several academies.

To astronomy, Gauss was drawn almost by accident: on the 1st of January, 1801, the first observations of the asteroid *Ceres* were made. The observations were not numerous and they were inaccurate (as the apparent movement of the asteroid was only  $3^\circ$ ), the task of determining the orbit of the new celestial body seemed to be impossible. Nevertheless, Gauss, an exceptionally gifted computer, managed the task with success. In the course of the work, Gauss also developed a method to determine the orbit of a celestial body on

the basis of only few observations. Gauss's main astronomical treatise, *Theoria motus corporum cælestium* from 1809 contains much material relevant to mathematical statistics; the name *Gaussian curve* reminds us of this side of Gauss's production.

Gauss participated in many ways in purely practical activities. He planned and directed the triangulation of the electorate of Hannover with its heavy field work – the terrain in Hannover is rather flat, and as such unsuitable for triangulation. But Gauss combined practical geodesy with theory. In the 1820's he made fundamental studies of the differential geometry of surfaces. It was he who directed attention to the intrinsic properties of surfaces, not depending on the surrounding three-dimensional space. One such property is the total curvature of *Gaussian curvature* of a surface.

In his later days, Gauss became interested in physics, magnetism in particular. The unit of the density of magnetic flow is a gauss. Gauss and his physicist colleague *Wilhelm Weber* (1804–91) built in 1833 the first working electrical telegraph between Weber's laboratory and the observatory of Gauss. The distance was 3 km. The *Theorem of Gauss* in three-dimensional integration which relates two- and three-dimensional integrals is related to Gauss's investigations in magnetism.

Notes which Gauss did not publish show that he had made significant advances towards the theory of complex analytic functions. He knew the important double periodicity property of the inverses of elliptic integrals, the *elliptic functions*, and the *Cauchy integral theorem* which states that the integral of a complex analytic function along a closed path in the complex plane vanishes. What Gauss did publish was a study of the *hypergeometric series*

$$F(x; a, b, c) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 \\ + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3 + \dots$$

The study contains the first actual proof that a series converges.

Gauss's correspondence and his posthumous notes indicate that he was aware of the fundamentals of *non-Euclidean geometry* before Bolyai and Lobatchevsky. Gauss tried to study the geometric structure of space also empirically by measuring the angle sum of large triangles. Considering the errors, these measurements did not give any other result than that the sum is  $180^\circ$ .

## 11 Analysis becomes rigorous in the 19th century

One can think of the 18th century as a period of wild invention in mathematical analysis. The methods worked and that was enough – there was little question after the logical foundations. At the onset of the 19th century, more critical research attitudes emphasizing rigor started to gain ground. A better understanding of complex numbers led to their widespread use and towards the birth of complex function theory. – We discuss the development of analysis in terms of some mathematicians whose influence on it was central. Many of them were prominent in other fields of mathematics, too.

### 11.1 Cauchy and Bolzano

The pioneer of rigour in analysis is (together with Gauss) the Frenchman *Augustin Cauchy* (1789–1857). He was educated in the *Écoles Polytechnique*, he started his career as an engineer but soon moved over to mathematics. He worked as a teacher in the *École Polytechnique*. The textbook he wrote for this institute, *Cours d'analyse* (1821), is based on a definition of limit quite close to the modern one, although without the later ubiquitous  $\delta$  and  $\epsilon$ . The convergence of series is carefully examined. The limit was defined verbally:

”If consecutive values of a variable approach a fixed value endlessly in such a way that they finally devi-

ate from the latter arbitrarily little, the fixed value is denoted the *limit*.”

Cauchy defined continuity by demanding that the variable  $f(x + \alpha) - f(x)$  becomes arbitrarily small as the variable  $\alpha$  endlessly diminishes. In the case of a function of several variables Cauchy was mistaken. He thought that a function continuous with respect to each of its variables is itself continuous. Cauchy proved the intermediate value theorem of continuous functions, now known as *Bolzano's theorem* by constructing two sequences  $(X_n)$  and  $(x_n)$ , one decreasing and one increasing, such that  $x_n < X_n$ ,  $f(x_n)$  and  $f(X_n)$  of opposite sign and  $X_n - x_n = \frac{1}{m}(X_{n-1} - x_{n-1})$  (we divide  $[x_{n-1}, X_{n-1}]$  in  $m$  equal segments; at the endpoints of at least one of them  $f$  has opposite signs unless  $f$  has a zero); the sequences converge to a common limit and continuity guarantees  $f(x) = 0$ .

The systematic investigation of the convergence of series and the understanding of the importance of convergence is very much due to Cauchy. He gave the ratio and root tests as well as the product rule, which has its name from him. Cauchy tried to prove the validity of Newton's binomial series as follows. If  $\phi(a) = 1 + ax + \frac{a(a-1)}{2!}x^2 + \dots$ , then the product rule gives  $\phi(a+b) = \phi(a)\phi(b)$ . But Cauchy solved this functional equation for continuous  $\phi$  and got as solutions functions  $\phi(a) = \phi(1)^a$ . As  $\phi(1) = 1 + x$ , then  $\phi(x) = (1+x)^a$ . The binomial series was finally fully satisfactorily treated by Abel in 1826.

The differential  $dy$  of a function  $y = f(x)$  was for Cauchy the number  $f'(x)dx$ , where  $dx$  is a finite number. The integral of a function was for him not any more an antiderivative. He defined the integral  $\int_a^b f(x)dx$  as the limit of sums

$$S_n = (x_1 - a)f(a) + (x_2 - x_1)f(x_1) + \dots + (b - x_n)f(x_n),$$

as the lengths of the subintervals  $(x_i, x_{i+1})$  of  $(a, b)$  tend to zero. Cauchy was not familiar with the concept of uniform continuity, and he claimed that his definition works for all continuous  $f$ . To obtain the relation between this kind of an integral and an antiderivative, Cauchy proved his *intermediate value theorem* of differential calculus,  $f(x) - f(y) = f'(\xi)(x - y)$  for some  $\xi$  between  $x$  and  $y$ . The special case  $f(x) = f(y)$  had been given by *Michel Rolle* (1652–1719) more than 100 years earlier. Cauchy also introduced the generalized intermediate value theorem

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

The Cauchy general convergence condition was essential in many of his arguments. A necessary and sufficient condition for the convergence of a sequence  $(a_n)$  is the smallness of  $|a_{n+p} - a_n|$  for large  $n$  and all  $p$ . The condition was also observed by the Bohemian cleric *Bernhard Bolzano* (1781–1848), whose production, in advance of his time, was largely unknown to his contemporaries. Bolzano published already in 1817 – before Cauchy – the modern definition of continuity:  $f$  is continuous, if its variation is such that  $f(x+\omega) - f(x)$  can be made smaller than any given quantity by making  $\omega$  as small as needed. – A rigorous prove of the sufficiency of the Cauchy criterion is not possible without a rigorous definition of a real number.

Cauchy evidently did not quite understand the meaning of *uniform convergence* although his later works give some hints towards it. The first definition of this concept was given by the English physicist *George Stokes* (1819–1903).

Cauchy is (again with Gauss, whose results were not available to his contemporaries) the founder of the *theory of functions*, or the study of complex-valued functions of a complex

argument. In hydrodynamics, Euler and d'Alembert had already come across the system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

but it was Cauchy who recognized the significance of these equations, later known as the *Cauchy–Riemann equations* to the differentiability of a function  $w(z) = u(z) + iv(z)$  of a complex variable  $z = x + iy$ . Cauchy investigated integrals of differentiable complex functions of *analytic functions* along plane curves; in 1825 he published the *Cauchy integral theorem*, which states that the integral of an analytic function taken along the boundary curve  $\mathcal{C}$  of a simply connected region always vanishes. The theorem is – at least for functions with a continuous derivative – a simple consequence of the *Green formula* which relates curve and plane integrals:

$$\begin{aligned} \int_{\mathcal{C}} f(t) dt &= \int_{\mathcal{C}} (u + iv)(dx + idy) \\ &= \int_{\mathcal{C}} (udx - vdy) + i \int_{\mathcal{C}} (vdx + udy) \\ &= \iint_G \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) dx dy + i \iint_G \left( \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \right) dx dy = 0. \end{aligned}$$

Six years later Cauchy proved that an analytic function can be represented as a power series whose radius of convergence is the distance of the expansion centre and the closest singularity point. This again is a consequence of the *Cauchy integral formula*, valid for an analytic function:

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{w - z} dw.$$

Here  $\mathcal{C}$  is an arbitrary closed curve winding once around  $z$ . The power series expansion comes from writing the denominator as  $(w - z_0 - (z - z_0))$  and using the sum formula

of a geometric series. Lagrange's vision of representing all functions as power series turned out to be partially true.

After Euler, Cauchy is one of the most productive mathematicians ever. His articles cover almost all branches of mathematics. For instance the product rule of determinants and the theory of determinants altogether come mostly from him. In the theory of ordinary and partial differential equations, his contribution is considerable. The great number and volume of Cauchy's works compelled the French Academy of Science to adopt a maximal length, four pages, of articles published in its *Comptes rendus*. The revolution of 1830 made Cauchy, a catholic and political conservative, to leave France. He exile took him to Prague, among other places. However, there are no direct proofs of his possible contact with Bolzano.

## 11.2 Abel, Jacobi, Dirichlet

*Niels Henrik Abel* (1802–29), born in Norway on the island Finnøy near Stavanger, is one of (fortunately not numerous) first class mathematicians with a tragically short life. He was poor and arrived from the periphery of Europe. This may be one of the reasons for the unkind reception he got several times in the centres of science. Gauss did not reply his letters and the French Academy declined to receive his manuscript "because the handwriting was unintelligible". Abel earned his living as a private tutor. The news of Abel's appointment to a university chair in Berlin only reached Norway after his death. He died of tuberculosis.

Abel was among the first to realize the importance of convergence when infinite series are used ("not a single series in mathematics has been rigorously proved to converge!"). However, he is most famous for putting an end to the attempts made ever since the times of Cardano to find meth-

ods of solving algebraic equations algebraically. Abel proved at the age of 19 that a general method to solve a fifth degree equation cannot exist. A slightly deficient proof had been given before by the Italian physician *Paolo Ruffini*, 1765–1822, an able amateur mathematician.

Abel studied elliptic integrals and discovered the two periods of the inverse function, a fact that escaped Legendre. The same observation was made independently of Abel by Gauss and *Carl Jacobi* (German, 1804–51). The latter developed the theory of these inverses, the *elliptic functions*, further. Abel's starting point was the observation that the integral

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x$$

which looks similar to the elliptic integrals, is simplest to treat via its inverse  $x = \sin u$ . Jacobi introduced the notations  $\operatorname{sn}$ ,  $\operatorname{cn}$  and  $\operatorname{dn}$  for the elliptic functions, notations that emphasize the relationship to the trigonometric functions. The elliptic functions turned out to be periodic like their trigonometric counterparts, but there are two independent basic periods whose ratio is not real. Abel tried to get help from Cauchy in his problems with complex numbers.

Abel also paid attention to integrals

$$\int \frac{1}{\sqrt{P(x)}} dx,$$

where  $P$  is a polynomial of degree higher than 4, and their inverses or *Abelian functions*. Jacobi showed that such inverses can also be studied if there are more than one variable.

Without doubt, Abel is the most outstanding mathematician born in Scandinavia.

To Jacobi's merit belongs the understanding and systematic development of the *functional determinant* or *Jacobi determinant* of a system of  $n$  functions of  $n$  variables, a concept introduced by Cauchy. Jacobi considered even ordinary determinants as determinants of a system of  $n$  linear functions of  $n$  variables.

The German, even if of French roots, *Peter Lejeune Dirichlet* (1805–59) followed Gauss in the mathematics chair in Göttingen. His posthumous lectures in number theory popularized and supplemented Gauss's difficult *Disquisitiones Arithmeticae*. Dirichlet's best-known number theoretic result states that for any relatively prime  $a$  and  $b$ , the arithmetic sequence  $a_n = an + b$  contains infinitely many primes. One of Dirichlet's discoveries is the manifold usefulness in number theory of the simple *box principle* or *pigeonhole principle* (if  $n + 1$  objects are placed in  $n$  boxes, then at least one box contains at least two objects).

In analysis, Dirichlet developed the theory of Fourier's trigonometric series. He was the first mathematician to seriously investigate the convergence of these series. Dirichlet is considered to be the first one to present the modern definition of a function. In his article on Fourier series in 1837 he writes

If the variable  $y$  is related to the variable  $x$  in such a manner that any time a numerical value is assigned to  $x$ , there is a rule according to which  $y$  assumes a unique numerical value, the  $y$  is said to be a function of  $x$ .

In fact this passage is given in the context of the definition of a continuous function, and the essential point is that the function need not be defined by the same expression throughout its domain of definition. Already some years earlier, the Russian mathematician *Nicolai Lobatchevsky*

(1793–1856), the inventor of non-Euclidean geometry, had defined a continuous function with approximately the same words. But even before that, in 1829, Dirichlet had given an example of a function with no analytic expression: for rational  $x$ ,  $y = c$ , and for irrational  $x$ ,  $y = d \neq c$ .

Dirichlet proved that for a moderately regular function  $f$ , the Fourier series of  $f$  usually converges to  $f$ , and in points of discontinuity, if one-sided limits exist, the series converges to their average. – The concept of conditional convergence of a series comes from Dirichlet.

The *Dirichlet problem* is the central problem of potential theory: it asks for the continuation of a function defined on the boundary  $\partial G$  of a domain into the interior of that domain in such a way that the continuation satisfies the Laplace differential equation. In his discussion of the problem Dirichlet stated a somewhat deficient variational principle, known as the *Dirichlet principle*. It was to play a prominent part in the development of function theory in the latter half of the 19th century. According to the principle, a solution  $f_0$  of the Dirichlet problem minimizes the integral

$$\int_G |\nabla f|^2 dV$$

among all extensions  $f$  into  $G$  of functions coinciding with  $f_0$  on  $\partial G$ . Dirichlet and other users of the principle did not address the question of the existence of the minimum. The matter was finally cleared by Hilbert in 1899.

### 11.3 Riemann

In Göttingen – arguably the most important mathematical research centre in the 19th and early 20th centuries – Dirichlet’s successor was *Bernhard Riemann* (1826–66), an original mathematical thinker whose impact on the develop-

ment of modern mathematics was deep. Like Abel, Riemann died of tuberculosis at a relatively early age.

Riemann's thesis (1851) dealt with functions of a complex variable. It contained, among other things, the *Riemann mapping theorem*, which states that every simply connected plane domain can be mapped one to one and conformally, or microscopically as a similarity, onto any other such region by an analytic function. A still more revolutionary idea was the removal of the generally one to many character of algebraic and other analytic functions by defining them not on a plane region but on a many-sheeted surface lying on the region. The theory of *Riemann surfaces* which originated from this idea led to a many-faceted interaction of analysis and topology and was in fact crucial for the development of topology into an independent branch of mathematics.

Riemann's achievements in analysis include the *Riemann integral*, a more versatile tool than Cauchy's integral. Riemann arrived at his version of the integral when working with Fourier's series, where integration of discontinuous functions came up. Riemann's basic idea was to replace  $f(x_i)$  in the Cauchy sums  $\sum f(x_i)(x_{i+1} - x_i)$  by an arbitrary value  $f(\bar{x}_i)$ , where  $x_i \leq \bar{x}_i \leq x_{i+1}$ . The integrability condition then is that the sums  $\sum O_i(x_{i+1} - x_i)$ , where  $O_i$  is the total oscillation of  $f$  on the interval  $[x_i, x_{i+1}]$ , tend to zero as the division becomes finer. A uniformly continuous function turn out to be integrable. (Uniform continuity was rather hazily understood at Riemann's time.) On the other hand, Riemann gave an example of a function which had an infinite number of discontinuities but was integrable anyway. The standard way of representing Riemann integrals now, with upper and lower sums  $\sum m_i(x_{i+1} - x_i)$ ,  $\sum M_i(x_{i+1} - x_i)$ ,  $m_i = \inf_{x_i \leq x \leq x_{i+1}} f(x)$ ,  $M_i = \sup_{x_i \leq x \leq x_{i+1}} f(x)$ , of a function on a subdivision of the interval of integration, is due to *Gaston Darboux* (French, 1842–1917). – In his lectures,

Riemann also presented an example of a continuous function which is nowhere differentiable. (The first to construct such a function, deeply shattering the function concept of the early 19th century was Bolzano, but this result shared the fate of his other achievements: it remained unknown in the centres of science.)

The most famous open problem in mathematics is the *Riemann hypothesis*. Riemann conjectured that all non-real zeros of the function

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + \dots = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1},$$

of a complex argument  $s = \sigma + i\tau$ , where  $p_n$  is the  $n$ :th prime, the *Riemann  $\zeta$ -function*, lie on the line  $\sigma = \frac{1}{2}$ . Riemann's hypothesis has not yet been proved and no counterexample found. Riemann's hypothesis has many interesting consequences in number theory. In their proof of the prime number theorem, Hadamard and de la Vallée Poussin used the property  $\zeta(1 + i\tau) \neq 0$ .

Riemann was a mathematical genius, but he was not an apostle of mathematical rigour. Geometric-physical intuition was often decisive for him. For instance, Riemann's proof of his (quite correct) mapping theorem was based on the not so well established Dirichlet's principle. – We shall return to Riemann's work in pure geometry later on.

## 11.4 Weierstrass

The central figure in mathematical analysis in the latter half of the 19th century was *Karl Weierstrass* (1815–97). His road to the top was not straight. After unsuccessful law studies Weierstrass obtained a degree entitling him to teach mathematics in high school, and he taught mathematics in a small town until 1854, when the mathematical world found

him. The rest of his life Weierstrass spent in Berlin, in the Technical University and the University of Berlin.

Weierstrass' attitude towards mathematics was in a sense opposite to that of Riemann. Weierstrass tried to relieve analysis from all intuition, to establish it on a firm arithmetic basis. It was just Weierstrass who reminded Riemann of the deficiency of the Dirichlet principle. Weierstrass' program, the *arithmetization of analysis* was carried on by a large number of students, and the origin of several results published by them was actually the master himself.

Weierstrass carried to completion the strengthening of the foundations of the calculus by fully observing the importance of uniform convergence e.g. when different limit processes are commuted. The " $\epsilon, \delta$ "-technique was established by the Weierstrass School. In his lectures in the Berlin Technical University in 1861 Weierstrass gave the definition of continuity as follows:

... if it is possible to determine for  $h$  such a bound that  $\delta$  that every value of  $h$ , smaller in absolute value than  $\delta$ ,  $f(x + h) - f(x)$  is smaller than an arbitrary quantity  $\epsilon$ , which can be arbitrarily small, then infinitely small variations of the argument correspond to infinitely small variations of the function values...

Weierstrass left a lasting footprint in complex function theory. As the starting point of the theory of analytic functions Weierstrass took the power series. The central tool in function theory will then be *analytic continuation*: the domain where the power series representation of a function is valid is extended by redeveloping the series around a new centre. The convergence circle of the new series usually extends outside the convergence circle of the original series. By analytic continuation, any power series can be extended to a usually

many to one analytic configuration defined in a maximal domain. – Weierstrass' interest in complex analytic functions originated from his studies of elliptic and Abel functions which he made while teaching at high school. He published them in the annual report of the school.

### 11.5 The classes of irrational numbers and the rigorous definition real numbers

That real numbers can be divided into *rational* and *irrational* numbers was in principle known since the time of the Pythagoreans. Euler surmised in the 1740's that some real numbers might be *transcendental*, which means that they are not *algebraic* or zeroes of some polynomial with rational coefficients. The first concrete example of a transcendental number was given in 1844 by *Joseph Liouville* (French, 1809–82). According to Liouville, all numbers of the form

$$\frac{k_1}{10} + \frac{k_2}{10^{2!}} + \frac{k_3}{10^{3!}} + \dots,$$

where  $1 \leq k_i \leq 9$ , are transcendental. The proof rests on the fact that an algebraic number cannot be very closely approximated by rational numbers with bounded denominators. Later somewhat less exotic transcendental numbers have been presented, such as 0,1001000100001.... In 1873 *Charles Hermite* (French, 1822–1901) could prove that  $e$ , the Neper number, is transcendental. That the same is true for  $\pi$  was proved ten years later by *Ferdinand Lindemann* (German, 1852–1939). Lindemann's proof finally closed the problem of squaring the circle with Euclidean tools. There are lots of interesting numbers whose algebraicity or transcendence is still open: one of these is the Euler constant  $\gamma$  (which in fact is not even known to be irrational).

The central reason for many logical difficulties in mathematical analysis was the vagueness of the concept of *number*

itself. An irrational number could be understood to be the limit of a sequence of rational numbers, but, on the other hand, the definition of a limit presupposes the existence of a limit candidate. Cauchy and Bolzano had tried to define convergence of a sequence in terms of the numbers in the sequence alone (the *Cauchy convergence criterion*), and Bolzano tried to define real numbers as limits of sequences of rational numbers, but it was not until 1872 that such a definition could be satisfactorily made. The definition was given independently by *Charles Méray* (French, 1835–1911), who already earlier had paid attention to the controversy mentioned above, and by Weierstrass together with his student *Eduard Heine* (1821–81) and the latter’s collaborator *Georg Cantor* (1845–1918). The definition starts with the *fundamental sequences* or Cauchy sequences of rational numbers. Such a sequence (or an equivalence class of sequences) defines a real number. The real numbers inherit the order properties and arithmetic operations from the corresponding properties of the terms of the sequences. An essential thing is that every Cauchy sequence converges to a real number. The set of real numbers is complete, no further extension is needed.

Also in 1872 another definition of real numbers was given by *Richard Dedekind* (German, 1831–1916). It is more closely related to the geometric concept of number as a point on a line and also to the theory of proportions by Eudoxos. According to the definition, a real number is a *Dedekind cut*, a separation of the rational numbers in two distinct subsets  $A$  and  $B$  such that every number in  $A$  is smaller than every number in  $B$ . If  $A$  has a maximal element or  $B$  has a minimal element, the cut corresponds a rational number. Again, the same process applied to real numbers again produces real numbers: the set of real numbers is complete.

## 12 Geometry, 16th to 19th century

The progress in geometry in the 18th century was much slower than that of analysis. Geometry was not fashionable among the great mathematicians then. But in geometry, too, the 19th century was a period when the foundations were much cleared and quite new methods were developed. The different developments were often overlapping, and a consistent picture is hard to give from other than the events leading to the birth of non-Euclidean geometry.

### 12.1 The origins of projective geometry

The art theorists in the Renaissance had contemplated questions related to perspective, such as the transformations experienced by a figure when it is viewed from different directions. The real originator of *projective geometry*, the branch of geometry which essentially studies properties of figures that are preserved in projections, is considered to be the French architect *Gerard Desargues* (1593–1662). His work *Brouillon projet d'une atteinte aux événements des rencontres d'un cone avec un plan* (A sketch to treat the events when a cone and a plane meet, 1639) did not get much attention (only 50 copies were printed). Desargues used an original botanical terminology to introduce concepts typical in later projective geometry, such as infinitely distant points where parallel lines meet. Desargues' name is preserved in the *theorem of Desargues* which states that the extensions of corresponding sides of triangles in a "perspective position" (the lines through corresponding vertices meet at a common

point) intersect at points lying on a line. Desargues derived a great number of general properties of conic sections using the invariance in projections of the so called harmonic point sets. Theorems which are easy to prove for circles can be extended to general conic section by this technique.

Desargues had a single student, Blaise Pascal. Pascal proved already at the age of 16 that the extensions of the sides of a hexagon inscribed in a conic section meet on a line. Pascal used a projective way of thinking. His theorem is relatively easy to prove for a circle. Pascal inferred that the intersection properties are preserved in projections which validates the theorem in the general case. Pascal derived 200 different corollaries from his theorem.

The synthetic methods of projective geometry were all but forgotten when analytic methods formed the mainstream of mathematics. The methods received new significance in the works of Gaspard Monge and the most important representative of the school he founded, *Jean-Victor Poncelet* (1788–1867). *Projective geometry* as an independent field of research was originated in the studies of Poncelet.

Poncelet was an engineer officer in Napoleon's Russian expedition and he was taken prisoner; he developed his geometric thoughts in when staying in Saratov as a war prisoner. Poncelet showed that in plane geometric statements one usually can interchange the words *point* and *line* without changing the truth value of the statement. Poncelet considered this principle to be a consequence of the reciprocity of polar lines and poles of a conic section. This *duality principle* was methodically applied by Poncelet's rival *Joseph Diaz Gergonne* (1771–1859). Poncelet's other geometric principle was continuity: the properties of "general" figures are preserved when they are continuously transformed into figures equally "general". Cauchy considered that this idea of "infinitely small changes" was inexact.

Poncelet supplemented the selection of geometric objects by ideal and imaginary ones (a line always intersects a circle, either at a real or an imaginary point) and by so doing furthered the abstractness of mathematics. – From Poncelet comes the beautiful *nine point circle* in elementary geometry: the feet of the altitudes, the midpoints of sides and the midpoints of the segments from the vertices to the orthocentre of a triangle all lie on a circle.

## 12.2 Synthetic and analytic geometry

The methods used by Poncelet were usually *synthetic*, not resorting to tools of analysis. In their purest form the synthetic methods appear in the works of *Jakob Steiner* (1796–1863), Swiss-born but mainly active in Berlin. Among other things, he discovered the importance of *inversion* or reflection in a circle. Steiner gave several solutions to the *isoperimetric problem*, the problem of the form of a curve of given length, bordering an area of maximal area. As a pure geometer, Steiner did not understand the arguments which claimed that his proof is only valid if the existence of the maximum is secured. Weierstrass gave a satisfactory solution to the isoperimetric problem using methods of the calculus of variations.

Poncelet and Steiner developed methods for solving Euclidean problems with a simpler set of tools than a straightedge and compasses. *Lorenzo Mascheroni* (Italian, 1750–1800) had proved in 1797 that Euclidean constructions can be done with compasses alone (assuming a line is drawn when two of its points have been obtained). Poncelet and Steiner showed that the constructions can be performed with a straightedge alone, provided one fixed circle and its centre are known. In 1927 a book with the name *Euclides danicus* published anonymously in 1672 by the Danish *Georg Mohr* (1640–97) and then forgotten was found. It

already contained Mascheroni's results.

In pure Euclidean geometry, some advances were made in the 18th century. *Giovanni Ceva* (Italian, 1647–1734) published a theorem unifying the results concerning the special points in a triangle, the *theorem of Ceva* in 1678. According to the theorem, segments  $AX$ ,  $BY$  and  $CZ$  joining the vertices of a triangle to points on the opposite sides are concurrent if and only if  $AZ \cdot BX \cdot CY = AY \cdot BZ \cdot CX$ . The name of *Robert Simson* (English, 1687–1768) is attached to the interesting *Simson line* of a triangle (the orthogonal projections of a point on the circumscribed circle of a triangle on the sides of the triangle lie on a line), although the line is first mentioned in print in 1797. Euler published his result on the *Euler line* of a triangle in 1765.

Geometry was studied from the analytical point of view by *Julius Plücker* (1801–68), among others. He discovered – concurrently with some other geometers – the useful *homogeneous coordinates*. When a point  $\left(\frac{x}{t}, \frac{y}{t}\right)$  in the plane was designated as the triple  $(x, y, t)$ , the special position of the point at infinity was removed and the duality of a point and a line became evident. The equation  $pu + qv + rw = 0$  represents all lines passing through the point  $(u, v, w)$  as well as all points on the line determined by the triple  $(p, q, r)$ . Plücker generalized the duality to space (where planes are duals of points and lines are duals of lines); the same programme was carried out by *Michel Chasles* (French, 1793–1880). Chasles initiated the use of *directed segments*, associated to vectors.

Analytic geometry was generalized to general  $n$ -dimensional space by the versatile English lawyer and mathematician *Arthur Cayley* (1821–95). Cayley used determinants as his tools. The equation of a line in the plane homogeneous

coordinates is

$$\begin{vmatrix} x & y & t \\ x_1 & y_1 & t_1 \\ x_2 & y_2 & t_2 \end{vmatrix} = 0,$$

and the corresponding object in  $n$ -dimensional space – a hyperplane passing through  $n$  points – can be defined by an analogous determinant with  $n + 1$  rows.

### 12.3 The birth of non-Euclidean geometry

The possible dependence of Euclid's fifth postulate or parallel axiom on the other axioms exercised the minds of many mathematicians in the course of two millennia. Simpler alternatives to the axiom were proposed and attempts were made, for instance by Ptolemy, Proclus, Nasir Eddin al-Tusi and Omar Khayyam to prove it as a theorem. In the 18th century, the Italian Jesuit mathematician *Girolamo Saccheri* (1667–1733) tried to give an indirect proof for the axiom. His starting point was a quadrilateral with two right angles and two equal sides. The parallel postulate is equivalent to the fact that the two other angles, which are equal in any case, are right angles. Saccheri assumed the angles to be acute or obtuse and derived consequences. The obtuse assumption indeed led to a contradiction, but the acute case remained inconclusive, although Saccheri thought to have arrived at a contradiction here too. Saccheri's work was not in vain: in fact he derives a great number of theorems of non-Euclidean geometry.

Work parallel to that of Saccheri was done by the Swiss-born *Johann Lambert* (1728–77) and Legendre. Lambert started from a quadrilateral with three right angles. Lambert could not rule out the possibility that the fourth angle might be acute, and he gave up the publication of his treatise on the parallel axiom. Lambert's friend *Johann Bernoulli*

III, however, printed the book *Die Theorie der Parallelinien* after Lambert's death in 1786. Lambert recognized the logical possibility of non-Euclidean geometry and in fact had a model for it: it would have been the surface of a sphere of negative radius.

The different editions of Legendre's popular textbook *Éléments de Géométrie* (1794) (which was considered to be a replacement for Euclid's Elements) contained extensive considerations of the parallel postulate. With the additional assumption of infiniteness of space Legendre could prove that the sum of the angles of a triangle does not exceed  $180^\circ$ , and if a single triangle with angle sum  $180^\circ$  exists, then all triangles have the same angle sum, and the parallel postulate holds. The last edition of the book edited by Legendre appeared in 1833.

The famous work *Kritik der reinen Vernunft* (1781) of the philosopher *Immanuel Kant* (1724–1804) divided truths into two categories: the self-evident *a priori* ones and the *a posteriori* ones obtained by experience. To Kant, Euclidean geometry with its parallel axiom stood for a priori truth, basic truth anchored to the mind of man by its nature.

In the 1810's, Gauss became convinced of the possibility of replacement of the parallel postulate by some other assumption without destroying the system of geometry. Because he never published his ideas, non-Euclidean geometry is attributed to two mathematicians who worked outside the established centres of learning. Both inventors had some connections to Gauss. One of them was the Russian *Nicolai Ivanovitch Lobachevsky* (1792–1856), the "Copernicus of geometry", the rector of the University of Kazan (800 km east of Moscow). A teacher of Lobachevsky had been *Martin Bartels* (1769–1833; later in the University of Tartu). Lobachevsky had believed to possess a proof for the parallel postulate and he even published it, but at some time be-

tween 1826 and 1829 he realized that a consistent system of geometry can be constructed on the assumption that several lines can pass through a point not on a given line such that they do not intersect it. The distribution of Lobachevsky's publications was slow, but they eventually reached Gauss. Although Gauss acknowledged Lobachevsky's results in a private letter, he never commented them in public. In a letter to the astronomer Bessel Gauss told that he refrained from publication of his own results in fear of the "outcry of the Boeotians", possible public ridicule.

The other inventor of non-Euclidean geometry was the Hungarian officer *János Bolyai* (1802–60), born in the Transylvanian town Koloszvár, now Cluj in Romania. His contact to Gauss was his father *Farkas (Wolfgang) Bolyai*, a mathematics teacher who had studied in Göttingen together with Gauss. Farkas Bolyai had tried to prove the parallel postulate, and his futile attempts made his attitude quite negative towards his son's interest in the problem. Nevertheless, the son succeeded in construction a geometry where infinitely many different parallels to a given line pass through a given point. János Bolyai's study was evidently written in 1823, but it was published in 1832 as an appendix *Appendix scientiam spatii absolute veram exhibens*, of 26 pages, in Farkas Bolyai's textbook *Tentamen Juventutem studiosam in elementa matheseos puræ introducendi* (Lobachevsky published his theory in Kazan in 1829 in Russian and in 1840 in Berlin in German). Gauss again refused to comment: "Should I praise this work I should praise myself, because I have had similar thoughts for several years". This acknowledgement and refusal of priority and the appearance of Lobachevsky's work in German depressed Bolyai to the extent that he never more published anything substantial.

The basic idea in the geometry of Gauss, Lobachevsky and Bolyai is that given a point  $P$  at a distance  $a$  of a line  $AB$ ,

there are lines through  $P$  such that they meet  $AB$  and lines such that they do not meet  $AB$ . If  $C$  is a point on  $AB$  and  $CP \perp AB$ , then there is an angle  $\alpha$  such that lines through  $P$  meeting  $CP$  at an angle at least  $\alpha$  do not meet  $AB$  or are parallel to  $AB$ . The angle  $\alpha(a)$  is determined by the equation

$$\tan \frac{\alpha(a)}{2} = e^{-a}.$$

The first to understand the real significance of non-Euclidean geometry was Riemann. His famous habilitation thesis, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* (1854), whose subject was suggested by Gauss, assumed a quite abstract position towards geometry. His work contained a far-reaching program: the subject matter of geometry is not the set of points, lines and planes in space but general  $n$ -dimensional *manifolds*, whose properties are determined by a metric which can vary from point to point. The distance is determined by a differential formula

$$ds^2 = \sum g_{ij} dx_i dx_j,$$

where  $g_{ij} = g_{ji}$  are functions that make  $ds^2 > 0$ . The Riemannian concept of space forms the basis of the general theory of relativity. – Riemann presented a simple model for a geometry where all lines intersect: the sphere and its great circles. Later *Eugenio Beltrami* (Italian, 1835–1900) found an analogous model for the geometry of Lobachevsky and Bolyai. It was the *pseudosphere*, a surface of revolution generated by the tractrix curve.

Non-Euclidean geometry liberated geometry. Different geometries constructed on different sets of axioms were possible, and the question of the geometry of the real world was turned over to physics.

## 12.4 The foundation of Euclidean geometry

The parallel postulate is not the only part in Euclid's system subject to criticism. Euclid had considered a concrete world based on observation. Gradually a new thinking emerged, according to which fundamental concepts such as a point and a line are idealizations of observational facts, but as far as axioms are considered, they are undefined objects. The construction of the system of geometry could in no point rest on the observational evidence we have on these matters. In particular, the concepts "betweenness" and "congruence of figures" were vague or had somewhat circular definitions in Euclid's system.

A significant new axiomatization of geometry was presented by *Max Pasch* (German, 1843–1930). In 1882 he gave an axiomatization of projective geometry, but its ideas were as well applicable to Euclidean or non-Euclidean geometry. To Pasch, the concepts point, line, plane and congruence of line segments were the undefined basic concepts, and axioms were those statements concerning these fundamental concepts which could be used in proving theorems. Later on, several alternative versions of the axioms of projective geometry were presented.

New axioms for Euclidean geometry were presented by *Giuseppe Peano* (Italian, 1858–1932) in 1891 and *David Hilbert* (German, 1862–1943) in 1899. Hilbert's system was given in his book *Grundlagen der Geometrie* which has appeared in many revised editions. Peano's undefined fundamental concepts were point, line segment and motion. Hilbert's system was close in spirit to Euclid, and it is the most widely accepted one. Hilbert's undefined concepts were point, line and plane. To emphasize their independence of observed world he asked his reader to mentally replace the words point, line and plane by the words table, chair and

mug.

The arbitrariness of the fundamental objects and axioms raised the question of the mutual compatibility or consistency of the axioms. The question was relevant in the context of non-Euclidean geometry, but there a reduction by Euclidean models to Euclidean geometry had settled the matter. Hilbert proved the consistency of his system by constructing an analytic model, based on real numbers. Since real numbers could be built on natural numbers, the consistency of geometry rested on the consistency of arithmetic.

## 12.5 Klein and the Erlangen programme

*Felix Klein* (1849–1925) was the central organizer in German mathematics at the end of 19th and beginning of 20th century. He started his career as Plücker's assistant. The main part of his work was done in Göttingen.

In the 1870's, Klein occupied for some time the chair of mathematics in the university of Erlangen. His inauguration talk of 1872 there is known as the *Erlangen programme*. Klein observed that a *group*, born in algebra, was a suitable structure for describing different geometric systems. Each geometric system is characterized by a group of transformations or bijective mappings of the underlying set. The investigation of the system means finding out properties which remain invariant in the transformations. The most general transformations for Klein were the projective transformations, in homogeneous coordinates

$$\begin{aligned}x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3.\end{aligned}$$

For instance the Euclidean metric geometry is equivalent to the group of translations, reflections and rotations, defined

by

$$\begin{aligned}x' &= r_1(x \cos \phi - y \sin \phi + a) \\y' &= r_2(x \sin \phi + y \cos \phi + b),\end{aligned}$$

$r_1, r_2 = \pm 1$ . The impacts of the Erlangen programme can still be observed in present-day geometry teaching.

The so called *hyperbolic model* of non-Euclidean geometry was discovered by Klein. In the model, a circular disc stands for the plane, circular arcs perpendicular to the circumference are the lines, and the circumference itself stands for the point at infinity.

Klein's central position in the mathematics at the end of the 19th century was for him a splendid starting point for writing on the history of mathematics in the 19th century. Klein's book *Entwicklung der Mathematik im 19. Jahrhundert* (1926–27), based on his lectures in Göttingen late in life, is one of the few treatises on the history of mathematics authored by a true first rate mathematician. It is a highly readable text even now.

## 13 Algebra in the 18th and 19th centuries

Up to the 19th century, the subject matter of algebra was the solution of polynomial equations  $P(x) = 0$  and the study of the properties of the solutions. The concepts central in present day university algebra courses were almost unknown in the 18th century.

### 13.1 The algebraic solution of polynomial equations

The results of Ferro, Tartaglia, Cardano and Ferrari in the 16th century left open the possibility of solving equations of degree  $n \geq 5$ . In the 17th century, *Ehrenfried Walter von Tschirnhausen* (1651–1708) could prove that by suitable transformations, any equation of degree five could be reduced to the form  $x^5 + ax + b = 0$ .

New deep insights into the solution problem were brought by Lagrange. His basic idea was to consider functions of the roots of an equation that give the same value for all or some permutations of the roots. If, for instance,  $x_1$ ,  $x_2$  and  $x_3$  are the roots of a cubic equation  $P(x) = 0$  and  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  is a complex root of the equation  $z^3 = 1$ , then  $(x_1 + \omega x_2 + \omega^2 x_3)^3$  only has two different values for the six permutations of the roots. This is related to the fact that the resolvent equation of a cubic equation is a quadratic equation. Lagrange's observations yielded a unified procedure for forming the resolvent equation of a given

equation. Lagrange's work was continued by Ruffini, who showed that Lagrange's methods could not produce resolvent equations of lower degree if the degree of the original equation is at least five. From this he drew the somewhat premature conclusion that a fifth degree equation is algebraically unsolvable.

A significant impact to the theory of algebraic equations was given by Gauss. In his book *Disquisitiones arithmeticae* he showed that the *cyclotomic equation* of circle division equation  $x^p - 1 = 0$ , where  $p$  is a prime, can be decomposed to a sequence of algebraic and algebraically solvable equations of lower degree; the degrees of these equations are factors of  $p - 1$ . So there are algebraically solvable equations of high degree.

Already as a high school student, Abel tried to apply Gauss's method to the general equation of degree five. He first believed to have succeeded in the solution, but soon discovered his mistake. Abel's proof of the insolvability of the fifth degree equation was based on his observation that a necessary and sufficient condition for solvability is that every root expression in the solution is a rational expression in the roots and certain roots of unity.

The final form of the theory of algebraic solvability of algebraic equations was given by *Evariste Galois* (French, 1811–32). Galois showed in 1831 that the roots of an algebraic equation can be expressed rationally in terms of the coefficients of the equation and their roots exactly when a certain subgroup, the *Galois group*, of the group of permutations of the roots has a property which we now call *solvability*. – Galois is one of the tragic and romantic figures in the history of mathematics. He twice failed the entrance examination of the *École Polytechnique*, he was sacked from the *École Normale*, his opinions were radical and he was in jail because of them and he died, before reaching the age of 21,

because of wounds obtained in a duel fought for a lady of doubtful reputation. His contemporaries had not much understanding to his writings, some of which were lost by the editors of journals. A decade after his death, his results were published. According to legend, Galois wrote his main studies on the night before the duel, but this is an exaggeration. But a letter written on that night to Galois' friend *Auguste Chevalier* (1806–68) contains an outline of Galois' programme.

### 13.2 "The liberation of algebra"

In a development parallel to the one in geometry, where Euclid's postulates and visible properties of the real world no more determined the subject, algebra started to consider systems deviating from the "given" systems of natural or real numbers. The indispensable negative and complex numbers needed to be understood. One of the first algebraists representing the more modern point of view was *George Peacock* (English, 1791–1858). In 1830 he published a textbook of algebra which tried to be logically satisfactory – algebra was earlier mainly considered to describe computation in a more abstract setting, and proof belonged to geometry only. Peacock proposed that there are two kinds of algebra, arithmetic and symbolic. The former codifies the rules for calculating with non-negative numbers while the latter operates with arbitrary quantities which need not be numbers. But the rules of computation are universal, because of the principle of "permanence of equivalent forms" or the preservation of the laws of operations. If two expressions are equivalent when non-negative numbers are substituted for the symbols, then they are equivalent also when they stand for arbitrary quantities. Properties such as *commutativity*, *distributivity* and *associativity* were recognized and named in the beginning of the 19th century.

Peacock's countryman *Augustus De Morgan* (1806–71), known for *De Morgan's laws* in logic and set theory, went one step further. Instead of quantities he manipulated fully abstract symbols with arbitrary content. De Morgan did not consider the possibility of systems obeying rules different from those of ordinary arithmetic. He compared systems based on axioms without real content to puzzles which are assembled upside down, without looking at the picture.

### 13.3 Hamilton and non-commutativity

An extension of algebra much more substantial than those of Peacock or De Morgan was performed by *William Rowan Hamilton* (Irish, 1805–65). Hamilton's treatise *Theory of Algebraic Couples* (1835) established the method of extending number systems by considering pairs of numbers. Hamilton defined negative numbers as pairs of positive numbers and rational numbers as pairs of integers. His step from rational to real numbers was not particularly convincing. But Hamilton defined complex numbers as pairs  $(a, b)$  of real numbers thus reducing the aura of mysticism which had surrounded them.

Hamilton tried stubbornly to generalize complex numbers to dimension three, but he had no success. But in 1843 he realized that if commutativity of multiplication is dispensed with, then the system of "four-dimensional" *quaternions*  $a + bi + cj + dk$  can be endowed with meaningful addition and multiplication if the rules  $ii = jj = kk = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$  and  $ki = -ik = j$  are obeyed. Because  $(ai + bj + ck)(di + ej + fk) = -(ad + be + cf) + (bf - ce)i + (cd - af)j + (ae - bd)k$ , the quaternion product includes both usual vector products, the scalar product and vector product.

By giving up commutativity, Hamilton took a major step to-

wards the abstractness and arbitrariness of algebraic structures. – Hamilton was a versatile scientist and his contributions range from optics and mechanics to graph theory. The state of a mechanical system can be described by its *Hamiltonian*, a generalization of energy, and the generalized spatial and momentum coordinates  $q_i, p_i$ . These satisfy the *Hamilton equations*

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$

The concept *Hamiltonian cycle* in graph theory, denoting a path visiting every vertex of the graph just once, is based on the game *Icosian* invented and marketed by Hamilton. In the game, a string has to be carried through the 20 vertices of a regular dodecahedron. The vertices carried the names of various cities, so in effect the problem was to travel around the world.

Hamilton was a precocious child with multifaceted talents. (At the age of 13 he could truthfully claim that he had learned a new language for each of the years he had lived.) He was one of the not so many first class mathematicians with excessive use of alcohol. And after he discovered the quaternions, he devoted all his energy to them. To some of his followers, the quaternions reached an almost the status of a religion. The theory was soon extended to comprise the *octonions*.

Another discoverer of non-commutative algebra was *Hermann Grassmann* (German, 1809–77), whose difficult *Ausdehnungslehre* (1844) presented an algebraic structure comprising vectors of different dimensions. Grassmann's ideas were simplified into the now usual three-dimensional vector algebra with its two products, one of which is non-commutative, by *James Clerk Maxwell* (1831–79) the English creator of electromagnetic theory and *Josiah Williard*

*Gibbs* (1839–1903), the American physicist. *Vector Analysis* by Gibbs was published in 1901. The more concrete vectors replaced the quaternions as physicist's tools.

The quaternions inspired several extensions of the number systems. Weierstrass proved in 1861 that the most general number system which has all the good properties of the real and complex number systems is just the system of complex numbers.

### 13.4 Matrices

*Matrix algebra* was also born at the middle of the 19th century. The non-commutative multiplication of matrices and the convention of denoting a sequence arranged in the form of an  $m \times n$  array by a single letter was invented in 1858 by Arthur Cayley when he considered the combination of two linear transformations

$$(x, y) \mapsto (ax + by, cx + dy), \quad (u, v) \mapsto (Au + Bv, Cu + Dv).$$

Gauss had already written down the combination as  $(x, y) \mapsto ((Aa + Bc)x + (Ab + Bd)y, (Ca + Dc)x + (Cb + Dd)y)$ . The theory of determinants had arisen before, independently of matrices. Cayley's innovation was that determinants should be denoted by two vertical lines,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The main interest of Cayley and his countryman *James Sylvester* (1814–97) was the study of *forms*, or homogeneous polynomials in two or several variables and their *invariants* or functions of their coefficients which remain unchanged when certain transformations of the variables of the polynomials are performed. (E.g. one invariant of  $ax^2 + 2bxy + cy^2$

with respect to a rotation is the discriminant  $b^2 - ac$ .) Sylvester established the method of transforming a form to the *principal axes* form. A special case of this is the transformation of the equation of a second degree curve or surface to the canonical normal form. – Like Cayley, Sylvester also worked as a lawyer as well as a mathematician. He was twice professor in Johns Hopkins University in Baltimore, thus being one of the earliest representatives of high level mathematics in America.

The word *matrix* was coined by Cayley. Its etymology is 'womb'; the meaning may have arisen from the rectangular framework used by compositors in a printing press. The word *invariant*, *discriminant* and *Jacobian determinant* were introduced by Sylvester.

The theory of invariants was an important field of study in the latter half of the 19th century. The question of the so called complete system of invariants was more important than finding invariants of separate polynomial types. In 1888 Hilbert solved the question by a higher level theorem which gave the existence of such systems as a corollary. Hilbert's result made much of the invariant theory irrelevant and interest in it waned.

### 13.5 Algebraic structures

Among the structures usually covered in first courses of algebra, the *group* is the oldest. The group concept first came up with the investigations into the possibilities of solving algebraic equations: the group properties appear implicitly already in the investigations of Lagrange and Ruffini. The word was first used in its present mathematical sense by Galois. Commutative groups are called *Abelian groups*. This comes from the interchangeable functions appearing in Abel's theory expressing different roots of an equation in

terms of one of the roots.

Groups, in particular *substitution groups* or groups of permutations, were studied by several mathematicians around the middle of the 19th century, among them Cauchy. The abstract definition of a group with the group operation defined by a table first appeared in an article of Arthur Cayley in 1854, but it remained largely unnoticed. In 1870, *Leopold Kronecker* (German, 1823–91) published the definition of an abstract (commutative) group as a novelty. The group was finally established in *Camille Jordan's* (French, 1838–1922) *Traité des substitutions*, which also appeared in 1870. At about the same time, *Sophus Lie* (1842–99, Norwegian, but active in Germany, too) noticed the value of the group structure in the theory of differential equations, and Felix Klein in geometry. The *Lien groups* are groups which have besides the algebraic structure also a topological one, such that the group operations are continuous functions. Lie groups are useful e.g. in the integration of partial differential equations.

The concepts *field*, *ring* and *ideal* were introduced by the Germans *Eduard Kummer* (1810–93), *Leopold Kronecker* (1823–91) and Richard Dedekind who all studied algebraic theory of numbers. The field was implicit in the work of Galois, however. The abstract axiomatic definition to these concepts was not given until the 20th century.

Kummer's goal was to prove Fermat's last theorem. Before him, the theorem was known to hold for the exponents 3, 4, 5, 7 and their multiplicities. Kummer succeeded in proving the claim for all exponents up to one hundred.

Kummer's student Kronecker was active in Berlin, first as an independent mathematician, then professor. He is remembered for his contributions to algebra but also for his rigid program for rigour. He only accepted results which could be reduced to properties of natural numbers. Thus

for instance the existence of irrational numbers was subject to doubt. ("The good God created the natural numbers, all the rest is the work of man.") Kronecker's attitude has a direct bearing to the concept of a field. If the existence of a useful number like  $\sqrt{2}$  is not guaranteed, its use has to be justified by a construction, which in this case is a field extension.

## 14 Mathematical logic and set theory

Analysis, geometry and algebra are all old and classical areas of mathematics. But quite new branches of mathematics were born in the 19th century. Among the most important of these are *mathematical logic* and *set theory*.

### 14.1 The birth of mathematical logic

Logic, the theory of the rules of inference as part of philosophy already was a well defined concept in antiquity. The central tool in Aristotelian logic was the *syllogism*, which consists of the "major premise", "minor premise" and "conclusion". ("All men are mortal. Socrates is a man. So Socrates is mortal.") The scholastics in the Middle Ages further developed Aristotle's system of logic. As algebra appeared as a support for geometry in the form of analytic geometry, thoughts about "computational logic" arose. An attempt was made by Leibniz. He tried to associate every primitive concept with a prime number; combination of concepts would be indicated by multiplication. If 3 stands for man and 7 for rationality, 21 would be a rational man.

The inventor of *mathematical logic*, logic detached from the meanings of natural language, is considered to be *George Boole* (1815–64). He was an English autodidact who supported himself and his parents as an elementary school teacher from the age of 16, but finally found his way to a professorship in Cork, Ireland. In 1847 and 1854 he published the treatises *Mathematical Analysis of Logic* and *Investigation of the Laws of Thought, on Which are Founded*

*the Mathematical Theories of Logic and Probability*. They give an introduction to *set algebra* or *Boolean algebra* and explain how syllogistic inferences can be translated to an algebraic language. As signs for set union and intersection Boole used the symbols  $+$  and  $\times$ , and his union was exclusive. Since syllogisms in fact are statements about sets and subsets, the Boolean algebra covered classical logic.

Boole made analogous innovations towards formalization and algebraization on other fields of mathematics, too: he introduced the *differential operator*  $D = \frac{d}{dx}$  and its symbolic manipulation into the theory of differential equations. In Bertrand Russell's opinion, "Boole invented pure mathematics". Boole's work was continued by Augustus De Morgan and *Benjamin Peirce* (American, 1809–80), who discovered the duality rule on the interchangeability of logical sum and product, known as *De Morgan's rule*. The graphical description of set algebra, the *Venn diagrams*, was invented by *John Venn* (English, 1834–1923).

## 14.2 The foundations of mathematics

Along with the emergence of mathematical logic and demands of increasing rigour in mathematics in general, it became desirable to reduce the foundations of mathematics, and arithmetic above all, to logic, which was viewed to be more "primitive" and reliable. There was a tendency to construct the system of mathematics deductively on the axioms of logic in the way Euclid had constructed geometry on his axioms and postulates. The fundamental question was that of essence of the natural numbers.

Richard Dedekind defined natural numbers in terms of an infinite set  $S$  and a successor relation in it. The set natural numbers is then the intersection of all those subsets of  $S$  that contain 1 and together with each element also

its successor. Another approach was tried by *Gottlob Frege* (German, 1848–1925), who tried to define natural numbers as equivalence classes of finite sets, the equivalence being defined by one to one correspondence. Frege is one of the inventors of symbolic logic. His *Begriffsschrift* of 1879 introduces the symbolism of predicate and quantor logic. Frege's main work is *Grundgesetze der Arithmetik* (1893–1903). Its purpose was to arithmetic and thus the whole of mathematics as an extension of logic. Just when the second part of the book was going to print, Frege got a letter from *Bertrand Russell* (English, 1872–1970) which informed him of a paradox which was fatal to his system. So Frege added a note indicating that the system he had created was fundamentally wrong. Frege never recovered from this blow.

Frege's work was continued by *Alfred North Whitehead* (English, 1861–1947) and Russell in their great *Principia Mathematica* (1910–13).

Giuseppe Peano created an original programme which aimed at presenting all mathematics in terms of a unified logical calculus. A part of this program is the *Peano axiom system* for natural numbers. It was published in 1889. According to Peano, the natural numbers  $N$  are a set characterized by the number 1, an injective successor relation  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that  $s(k) \neq 1$  for all  $k \in \mathbb{N}$ , and the induction axiom which say that for any  $S \subset \mathbb{N}$ , the conditions  $1 \in S$  and  $s(S) \subset S$  imply  $S = \mathbb{N}$ . [One can see from this Peano's opinion, and that of mathematics in more generality, to the ever upcoming pedagogical question whether 0 is an element of  $\mathbb{N}$ .] – Peano's impulses to analysis are considerable. His investigations to the existence of solutions of differential equations as well as his example of a space-filling curve are important. In 1903 Peano introduced an artificial language, *latino sine flexione*, and later returned to mathematics only occasionally.

### 14.3 Cantor and set theory

The problem of *infinity* had bothered in many ways the sophists of antiquity and the scholastics in the Middle Ages. Galileo Galilei had observed an essential feature of an infinite set, when he noticed that the square numbers could be associated in a one to one way with all natural numbers, although the former clearly comprised only a small fraction of the latter. Gauss and Cauchy thought that paradoxes like this indicated that actual infinity is not possible. But in his posthumous treatise *Paradoxen des Unendlichen* Bernhard Bolzano, who for instance proved with the help of a simple geometrical construction that the interval  $(0, 1)$  contains "as many" points as the interval  $(0, 2)$ , expressed the opinion that this property must be accepted. Finally Dedekind postulated that the defining characteristic of an infinite set is just the situation that the set has a proper subset which is in a one to one correspondence with the whole set.

Georg Cantor went much further than Dedekind in the classification of infinities. The first impulse to Cantor's thinking was given by problems arising in connection of the uniqueness of the Fourier expansions of functions discontinuous in a rather big set. Cantor's publications on set theory start in 1874. They introduce a hierarchical classification of infinite sets and the arithmetic of infinite cardinal and ordinal numbers. The most fundamental of Cantor's observations was that infinite sets need not be equipotent – as for instance Bolzano had thought. The question of the potency or cardinality of specific sets naturally comes up. Cantor proved that rational numbers can be arranged in a sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \dots,$$

which means that the sets of rational and natural numbers are equipotent. Cantor then showed that the algebraic num-

bers and natural numbers are equipotent. In fact, each polynomial with integer coefficients has a *height* which is the sum of its degree and the absolute values of its coefficients. For each height  $n$ , there is only a finite number of polynomials, they are of degree at most  $n$ , so there are only a finite number of algebraic numbers which appear as roots of polynomials of height  $n$ . Cantor called sets whose potency is at most the potency of natural numbers *numerable*; for the cardinality of a numerable set he introduced the symbol  $\aleph_0$ , *aleph-zero*, for the first letter in the Hebrew alphabet.

On the other hand, Cantor showed that there are "more" real numbers than natural ones. If  $(x_n)$  is a numerable sequence of real numbers, one can construct intervals  $I_n = (a_n, b_n)$ , with rational endpoints, such that  $I_1 \supset I_2 \supset I_3 \dots$  and  $x_n \notin I_n$ . The numbers which are in the intersection  $\cap I_n$  cannot be in the sequence  $(x_n)$ . Later on, Cantor proved the same result using his well known diagonalization procedure: taking any sequence of real numbers, one can construct a new one not in the sequence simply by taking the  $n$ :th decimal different from the  $n$ :th decimal of the  $n$ :th number in the sequence. The same idea can be utilized in proving that the collection of subsets of any set has a potency bigger than that of the basic set. Thus there is an infinity of infinities of different size.

Cantor also introduced the *continuum hypothesis* which states that no set exists having a potency between those of natural and real numbers. The hypothesis has neither been proved nor refuted. But results by *Kurt Gödel* (Austrian–Czech 1906–78) in 1938 and *Paul Cohen* (American, 1934–2007) in 1963 show that neither alternative contradicts the axioms of set theory.

Cantor's ideas met with considerable resistance. The opponents, of whose Kronecker carried most authority, were partly just prejudiced, but the vagueness of the concept

of a set gave rise to serious counter-arguments. Paradoxes arose in connection of very large sets. The most famous of these is *Russell's paradox* discovered by Bertrand Russell: the set whose elements are those sets  $S$  for which  $S \notin S$  is contained in itself as an element if and only if it is not contained in itself as an element. It became desirable to develop an axiomatic foundation of set theory in such a way that paradoxes or set causing them would be closed out. This program was carried out by e.g. *Ernst Zermelo* (German, 1871–1956), but Whitehead and Russell in their *Principia Mathematica* (1910–13) developed the *theory of types*, a classification of sets, for the same purpose. Zermelo formulated the useful *axiom of choice*, which has later turned out to be independent of the other axioms of set theory. – Cantor could not bear the heavy critique he encountered. He spent the last years of his life in a mental institution.

The simple or naive set theory used daily by mathematicians is essentially equal to Boole's algebra. In fact, Boole interpreted the elements of his algebra as sets.

## 15 Probability – recreation turned science

Earliest allusions to probability have been discovered in Talmud, the Jewish law. Certain rules for calculating probabilities of combined events appear there, to justify legal decisions. But probability as we know it has its origins in gambling.

### 15.1 Gambling and probability

Gambling easily connects with calculations of the likelihood of a win or ruin. Cardano wrote around 1526 a book *Liber de Ludo Aleæ*, Book of games with dice and there introduces the multiplication rule of probabilities. The book was published posthumously.

The starting point of probability theory is considered to be a letter to Pascal by *Antoine Gombaud* (1607–84), also known as *Chevalier de Méré*, with two problems on gambling. The first considered a game consisting of several matches. The stake is supposed to be given to the player first winning six matches, but the game has to be interrupted in a situation where one player has won five and his opponent three matches. What would be a just division of the stake? The question had been open for a long time. Luca Pacioli proposed the division 5 : 3, Tartaglia 2 : 1. Pascal and Fermat considered the problem in their correspondence and both arrived at the same solution 7 : 1, the former by a recursive argument based on Pascal's triangle, the latter by counting the elementary events. The other question considered a se-

ries of throws of two dice. How many throws there has to be for being more advantageous to bet for at least one pair of sixes in the series than not? There was a rule of thumb among gamblers that the number would be  $\frac{4}{6} \cdot 36 = 24$ , but empirical evidence did not support this.

Christiaan Huygens's booklet *De Ratiociniis in Aleæ Ludo* was published in 1657. Huygens considers expected value: if a game offers  $p$  possibilities of winning the sum  $a$  and  $q$  possibilities of winning the sum  $b$ , then one can with advantage invest at most  $x = \frac{pa + qb}{p + q}$  in the game. Applying this principle, Huygens showed that in a series of 24 double throws it is not advantageous to bet for a double six, whereas in a series of 25 throws it is.

Pascal applied a kind of expected value theory in theology: a man can live either as if there is a God or as if there is no God. In the latter case, it does not matter how you live. But if God exists, living right leads to salvation and living wrong to damnation. Because salvation is infinitely better than damnation, the expected value of living right is bigger than that of the other alternative, even if the probability of God's existence would be small.

## 15.2 Law of large numbers and the normal distribution

Jakob Bernoulli generalized Pascal's solution of the stake division problem to situations where win and loss are not symmetric. He thus in effect defined the general binomial distribution. If there are  $a$  possibilities of success and  $b$  possibilities of failure, the probability of  $k$  wins in  $n$  trials,

the probability of  $k$  wins was, according to Bernoulli,

$$\frac{\binom{n}{k} a^k b^{n-k}}{(a+b)^n}.$$

Bernoulli's *Ars Conjectandi* (1713) extended the probability concept from cases of throwing dice to everyday reality. In Bernoulli's opinion, absolute certainty is difficult to attain, but for *moral certainty*, it is sufficient to know that the supposed state of facts appears 999 times in one thousand. Bernoulli applied his theory of moral certainty to a situation where an unknown distribution is investigated by repeated observations. If there are  $r$  black and  $s$  white balls in an urn, and if  $X$  is the number of black balls lifted in  $N$  trials, then for large enough  $N$  the ratio  $\frac{X}{N}$  deviates from  $\frac{r}{r+s}$  less than  $\frac{1}{r+s}$  at most once in a thousand series on  $N$  trials. Bernoulli calculated that for  $r = 30$  and  $s = 20$ , it is sufficient to have  $N$  at most 25 550. That  $N$  was so large was a disappointment for Bernoulli. The application of the law of large numbers in practice was not very promising.

Abraham De Moivre sharpened the results of Huygens and Bernoulli. If the ratio of the possibilities of success and failure is  $a : b = 1 : q$ , then the number  $x$  of trials needed to have at least one success more likely than no success can be obtained from

$$\left(1 + \frac{1}{q}\right)^x = 2.$$

For large  $q$ ,  $x \approx 0,7q$ ; in the case of double sixes  $q = 35$  and  $x = 24,5$ . De Moivre developed the binomial probability formula by estimating the factorials. In the case of a symmetric binomial distribution he came up to the result that

the probability of  $\frac{N}{2} + t$  successes is approximately

$$p(t) = \frac{2}{\sqrt{2\pi N}} e^{-\frac{2t^2}{N}}.$$

The graph of  $p(t)$  approximates the binomial distribution. De Moivre found the points of inflection of the curve at  $\frac{1}{2}(n \pm \sqrt{n})$  and that a deviation of at most  $\frac{1}{2}\sqrt{n}$  from the centre took place with probability 0,682688. De Moivre did not in detail to asymmetric distributions. However, he found that Bernoulli's 25 550 trials could be reduced to 6498. – The *central limit theorem* in probability, which states that the distribution of the sum of almost arbitrary random variables approaches a normal distribution, if the number of variables increases, appeared in Laplace's works at the turn of the 19th century. The proof of the theorem with sufficiently loose assumptions only appeared in the beginning of the 20th century.

Probability theory was not very much applied to other real life cases than games of chance. The title of the fourth part of Bernoulli's *Ars Conjectandi* promised social, moral and economical applications, but it "only" gave the law of large numbers. Life annuities ( $A$  pays  $B$  a certain sum and  $B$  pays annually a certain sum to a person designated by  $A$ ) were usual in the 18th century but the sums were not computed on the basis of probabilities inherent in mortality tables; neither were lottery wins in proportion to the probabilities of winning.

### 15.3 Statistical inference

The first to seriously investigate the question of deducing the probability of an event from observations was the English clergyman *Thomas Bayes* (1702–61). He introduced

the concept of *conditional probability* and derived the formula

$$\frac{\int_a^b \binom{n}{k} x^k (1-x)^{n-k} dx}{\int_0^1 \binom{n}{k} x^k (1-x)^{n-k} dx}$$

to express the probability that the probability of success in a single trial is in the interval  $(a, b)$ , if one knows that a series of  $n$  trials produced  $k$  successes.

Although inference "from consequences to causes" is called *Bayesian*, the real pioneer of reverse probability was Laplace. In 1774 he published the result that  $k$  successes in  $n$  trials indicates that the probability of success in a single trial deviates from  $\frac{k}{n}$  at most by  $c$  is

$$\frac{2}{\sqrt{2\pi}} \int_0^{c/\sigma} e^{-\frac{u^2}{2}} du,$$

where

$$\sigma^2 = \frac{k(n-k)}{n^3}.$$

Laplace applied his result in practice, too. Between 1745 and 1770 251 527 boys and 241 945 girls were born in Paris. Laplace computed that the probability of a boy's birth is smaller than that of a girl would be  $1,15 \cdot 10^{-42}$ . It is morally certain that more boys are born than girls.

Another setting where statistical data required mathematical treatment was provided by astronomical observations. The determine the parameters of mechanical models on the basis of several observations led to situations where a number of not quite compatible equations had to be satisfied by not so many unknowns. In the 18th century, several solutions were proposed to find a satisfactory approximate

solution of the equations. Adrien-Marie Legendre published in 1805 as an appendix to a study of the determination of the orbit of a comet a discussion of the problem of finding a solution making a number of linear expressions of the unknowns as close to zero as possible. He gave balance arguments favouring a choice of the unknowns such that the sum of squares of the expressions would be minimal. The problem was easily solved by differential calculus.

Gauss published his version of the method of least squares in 1809. Gauss mentioned that he had used the method since 1795. Gauss gave a more thorough derivation of the method than Legendre. He first determined the distribution  $\phi$  of the observational error  $\Delta$ . Gauss could prove that its form is

$$\phi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2 \Delta^2}.$$

The derivation was based on the assumption that the average of the observations was the likeliest candidate for the correct value.

The use of the normal distribution was extended from the theory of observational errors to more general contexts by *Adolphe Quételet* (Belgian, 1796–1874). Quételet was originally an astronomer, but he observed the normal distribution in different statistics describing properties of man. An especially telling evidence for the distribution was a statistics showing the chest measure of 5732 Scottish soldiers; the measure was normally distributed around its average 40 inches. Quételet introduced the concept *probable deviation* for the interval on a normal distribution around the centre containing one half of the possible cases<sup>1</sup>

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<sup>1</sup> Quételet also invented the body-mass index weight/height squared.

*Francis Galton* (English, 1822–1911) tried to use Quételet's methods to prove Darwin's evolution theory. (Galton was Darwin's cousin.) Experiments on descendants of peas of various sizes led Galton to the concepts of *regression* and *correlation*. Galton inferred the size of correlation from the inclination of a line which approximately described the observed values.

The concept *root mean square deviation* as well as the use of least squares in determining the correlation coefficient were introduced into statistics in 1893 by *Karl Pearson* (English, 1871–1936). He also defined the partial correlation coefficients and introduced the *random walk* -theory. In 1900 Pearson presented the  $\chi^2$  test and the general pattern of testing a zero hypothesis on the basis of statistical data. Another widely useful test, the *t*-test or Student's test based on the distribution of the mean of a small sample, was developed in the beginning of 20th century by *William Gosset* (English, 1876–1937). Gosset contemplated the behaviour of the means of small samples when working as a chemist in a brewery in Dublin. It has been surmised that Pearson forced Gosset to use the pseudonym Student because as the editor of the journal *Biometrika* he did not want to publish an article of an author with a brewery as the background organization.

## 16 On mathematics in the 20th century

The 20th century produced mathematics and mathematicians of importance at least as much as all the previous centuries altogether. This is not surprising since the exponential growth of science implies (probably – an exact calculation might be challenging) that a majority of mathematicians of all times are alive now. The applications of mathematics have expanded as well as the needs for such applications. A full coverage of 20th century mathematics in any sense is not possible here.

### 16.1 Poincaré

The two outstanding figures in the mathematics at the turn of the 19th and 20th centuries are *Henri Poincaré* (French, 1854–1912) and David Hilbert.

Poincaré was one of the last generalists both in pure and in applied mathematics, probably the last universal mathematician. He did not stay long with one theme of research, but readily switched his interests. Poincaré was educated as a mining engineer in the *École Polytechnique* and *École des Mines*; as a professor in Sorbonne he lectured every year of a different subject and wrote books, also popular ones. According to different sources, the number of his scientific articles is something between 500 and 1500.

In pure mathematics, one of most enduring of Poincaré's achievements is the theory of *automorphic functions*, cre-

ated in the early 1880's. Automorphic functions, which can be viewed as generalizations of elliptic functions, are functions  $f(z)$  of a complex variable satisfying  $f(z) = f(g(z))$  for all members  $g$  of some group  $G$  consisting of transformations

$$g(z) = \frac{az + b}{cz + d};$$

the group has to satisfy the so called discontinuity condition. Poincaré called groups whose elements map the unit disc onto itself *Fuchsian groups* after *Lazarus Fuchs* (German, 1833–1902). More general discontinuous groups are *Kleinian groups* after Felix Klein. Poincaré and Klein had a fierce competition at the early stages of development the automorphic function theory. At the end, Klein experienced a nervous breakdown and never quite recovered his creative abilities. – The automorphic functions are connected with differential equations, non-Euclidean geometry and Riemann surfaces. For instance algebraic functions defined on a Riemann surface can be interpreted as functions defined in the unit disc, automorphic with respect to a suitable Fuchsian group.

Poincaré is one of the founders of algebraic topology and his contributions to differential equations are substantial. Like Laplace, he did important work in celestial mechanics, the  $n$ -body problem, and in physics otherwise, too. Some of his work anticipated Einstein's theory of relativity.

## 16.2 Hilbert and his 23 problems

Hilbert was an extremely versatile mathematician, too. He started his academic career in his birthplace, Königsberg. From 1896 he was professor in the University of Göttingen. Hilbert's first results were in the theory of invariants and in number theory. Hilbert gave a non-constructivist proof for the basis problem which had bothered invariant experts

for the last half of the 19th century. On reading Hilbert's article, the leading invariant theorist *Paul Gordan* (1837–1912) exclaimed that Hilbert's solution was theology, not mathematics. Hilbert got familiarized with number theory when he worked on the overview of the subject ordered by the *Deutsche Mathematiker-Vereinigung*, the German association of Mathematicians. The report compiled by Hilbert also unified and simplified earlier results. Hilbert's solution (in 1909) to the problem posed by *Edward Waring* (English, 1734–98) on writing an integer as a sum of  $n$ :th powers. Hilbert's axiomatization of geometry has been mentioned above. The convention of introducing real numbers axiomatically, without resorting to construction based on natural numbers, was introduced by Hilbert.

In the International Congress of Mathematicians in 1900, Hilbert gave a famous talk on the status of mathematics. In the talk he gave a list of 23 open problems on various branches of mathematics. The search for solutions of these *Hilbert problems* has occupied and still occupies mathematicians. Hilbert's first problem concerned the continuum hypothesis of Cantor, the second was about the consistence of the axioms of arithmetic and the sixth was about an axiomatization of physics. Some of the problems were quite specific in character, like the problem of irrationality or transcendence of the numbers  $2^{\sqrt{2}}$  or  $e^{\pi}$ , some very general, just outlining a direction of research. The problems came to be considered as a list of goals of mathematics, and any mathematician solving one of the problems automatically became famous. The first Hilbert problem which was solved was number three, asking for a proof for the fact that an arbitrary tetrahedron cannot be partitioned into a finite number of polyhedra such that could be recombined to form a cube. The solution was found in 1901 by Hilbert's student *Max Dehn* (1878–1952).

Besides the fields mentioned, Hilbert was active in integral equations, mathematical physics, and mathematical logic. The dubious Dirichlet principle was salvaged by him and returned to the collection of allowed mathematical tools. In the philosophy of mathematics, Hilbert was a representative of the *formalist school*. – Hilbert’s ”last words” addressed to those colleagues of him who doubted their ability to cope with the controversies in the foundations of mathematics are encouraging exhortation: *Wir müssen wissen, wir werden wissen!* (We must know and we will know!)

### 16.3 On German mathematics in the 20th century

Göttingen was the undisputed centre of German mathematics before World War 2. Of Göttingen mathematicians one can mention Klein, his successor *Richard Courant* (1888–1972), Hilbert’s student and successor *Hermann Weyl* (1885–1955) and *Emmy Noether* (1882–1935), algebraist and probably the most important female mathematician so far. Courant wrote and published in 1924 together with Hilbert the book *Methoden der mathematischen Physik*, which much influenced quantum mechanics, emerging in the 1920’s. The versatile Weyl published an axiomatization of Riemann surfaces in 1912. His impact on the general relativity theory of *Albert Einstein* (1879–1955) was considerable. Hilbert also was involved in the process of the discovery of general relativity. There is no dispute on the position of Einstein as the principal creator of the theory.

In 1933 the National Socialists assumed power in Germany. This had global consequences to the development of mathematics. An exodus of mathematicians of Jewish roots or thoughts not conforming to those of the Nazis started. Most of the emigrants found a place in the United States, among them Courant, Weyl, Einstein and Noether. The leadership

in mathematical quality and quantity obtained by America in the latter half of the 20th century is to a great part a consequence of this exodus.

In the Germany of 1930's, attempts were made to divide mathematics into the good Aryan and inferior Semitic ones. In this dubious process, some otherwise competent mathematicians participated. One of them was *Ludwig Bieberbach* (1886–1982), famous for his function theoretic conjecture on the size of the coefficients of the power series representation of univalent functions defined in the unit disc. (The conjecture was a good research project as it could be solved by various methods one coefficient at a time until the American *Louis de Branges* spoiled the fun by proving the entire conjecture in 1984.)

## 16.4 Topology

*Topology*, roughly speaking the study of properties of figures invariant in various continuous transformations is an organic part of much of modern mathematics. The origins of topology have been traced to Euler's solution of the Königsberg bridge problem, to Riemann's thesis or to Cantor's studies which lie in the terrain between set theory and topology. The term topology was first used by Gauss's student *Johann Benedikt Listing* (1806–82) in 1847. Listing's argument for replacing *Geometrie de Lage* (which is a translation of the Latin *analysis situs*, also denoting topology) by *Topologie* was that the former actually denoted projective geometry. Besides Listing, another student of Gauss, *Augustus Ferdinand Möbius* (1790–1868) studied the basic topologic properties of figures. Möbius invented *triangulation*, which proved to be a useful method. The topological properties of a polyhedron are reflected in the ways its surface can be divided into triangles.

The basic concepts of point set topology, an *open* and a *closed set* first appear in Cantor's works. The concept of convergence lies behind them. The first systematic presentation of topology was Poincaré's *Analysis Situs* in 1895. Poincaré was primarily interested in the combinatorial and algebraic aspects of topology. What he wanted to understand was the essence of the surface in four real dimensions defined by an equation  $f(x, y, z) = 0$  in complex variables. Poincaré studied general  $n$ -dimensional manifolds. He defined several characteristics describing their properties such as connectivity and introduced the fundamental group or homotopy group of a manifold. His assumption that a manifold possessing certain properties which belong to an  $n$ -dimensional sphere actually is a sphere was known as the *Poincaré hypothesis*. Somewhat surprisingly, the hypothesis could be proved for most  $n$ , but  $n = 3$  remained open until 2003.

*Felix Hausdorff* (German, 1868–1942) is regarded as the founder of point set topology. His *Grundzüge der Mengenlehre* (1914) introduces the axiomatic definition of a topological space in terms of the concept neighbourhood. Another important pioneer of topology is *L. E. (Luitzen Egbertus) Brouwer* (Dutch, 1881–1966). In 1911 he proved the central invariance property of open sets: no homeomorphism (one-to-one continuous mapping) exists between open sets in spaces of different dimensions. Brouwer also introduced *simplexes*, generalizations of triangles into  $n$  dimensions.

## 16.5 Real and functional analysis

The set theory of Cantor was the origin of new theories of *measurability* and integration. Among the pioneers in this field were *Emile Borel* (French, 1871–1956) (he was a mathematician with political ambitions; he was in the French parliament and even served as Navy Minister in the 1920's)

and *Henri Lebesgue* (French, 1875–1941). Borel restricted the sets to which measurability was relevant to sets which could be constructed by enumerable processes from intervals of the real line, the *Borel sets*. Lebesgue's thesis (1902) considerably extended Riemann's integral concept. His central idea was that instead of dividing the domain of the function to be integrated into very small sets (such that the value of the function is close to constant within the small sets), it is more advantageous to divide the range. Then it will be possible to integrate functions which are quite irregular, like Dirichlet's function which is continuous nowhere. On the other hand, one has to generalize the concepts of length, area and volume to sets which are irregular in the geometrical sense. The measure of Lebesgue works with a larger collection of sets than that of Borel.

Set theory, topology and measure theory all contribute to *functional analysis*. The starting points of functional analysis were the efforts to solve concrete *integral equations* arising in the calculus of variations and the Dirichlet problem and attempts at extending the concepts of analysis to the abstract spaces which had been created in set theory and topology. – The noun *functional* often denotes a function whose value is a number but argument a function; a transformation which attaches a function to a function is often called an *operator*.

Important pioneers in the study of integral equations were *Vito Volterra* (Italian, 1860–1940) and *Ivar Fredholm* (Swedish, 1866–1927). They tried to solve equations of the type

$$u(x) = f(x) + \int_a^b K(x, t)u(t) dt,$$

where  $f$  and  $K$  are known and  $u$  an unknown function. Volterra gave iterative solution to the equations and Fred-

holm approximated them with large systems of linear equations.

One of the first to use methods relying on an "infinite number of dimensions" with integral equations was Hilbert. His interest in the subject was aroused by a talk on Fredholm's investigations given by the Swedish mathematician *Erik Holmgren* in Göttingen in 1900. Hilbert did not himself define the closest infinite dimensional analogue of an Euclidean space, the complete inner product space with a numerable basis or *Hilbert space* (instead, he once had to ask his younger colleague Hermann Weyl about the meaning of the term). The axiomatic definition of a Hilbert space was presented in 1930 by *Johann (John) von Neumann* (Hungarian, later American 1903–57).

The study of abstract function spaces was initiated by *Maurice Fréchet* (French, 1878–1973). Fréchet's thesis of 1906 generalizes several concepts of analysis such as limit and continuity to general spaces. A special case of an abstract space was a *metric space*, whose definition was given by Fréchet, too.

An important event unifying and codifying functional analysis was the appearance in of *Théorie des opérations linéaires* by *Stefan Banach* (Polish, 1892–1945). The book gives the definition of one of the most central structures in functional analysis, that of a complete normed linear space or *Banach space* and derives the fundamental properties of this kind of a space.

## 16.6 Probability

In the beginnings of the 20th century, probability was transformed from being primarily a study of the games of chance into a central branch of mathematics. Its scope of applications widened from insurance and observation errors

to statistical mechanics – which counts among its pioneers Williard Gibbs, mentioned earlier – genetics and other biological sciences.

The influence of set theory and measure theory were felt in probability, too. Borel made the concept of probability more precise in his textbook in 1909, but a definitive axiomatization of probability was made in 1933 by *Andrei Nikolajevič Kolmogorov* (Russian, 1903–87). Kolmogorov, an extremely versatile mathematician and popularizer, reduced probability to measure theory. Another important Russian probabilist, *Andrei Andrejevich Markov* (1856–1922) published in 1906 and 1907 papers in probability chains, *Markov chains* thus initializing the study of *stochastic processes*. Markov chains describe systems with a number of possible states and random transitions from one state to another with the transition probability only depending on the state, not on the previous history of the system. – Markov also worked in number theory. As number theorist he is known as *Markoff*.

## 16.7 Unification tendencies

Modern mathematics has been differentiated in several branches. The standard count is that there are about 60 main fields, and each of the is again divided in subfields. To counter this splintering, attempts at creating syntheses were made in the 20th century. Already Peano aimed at unification. In Germany, the editors of the extensive *Enzyklopädie der mathematischen Wissenschaften* (1898–1921) had a similar goal.

The most remarkable synthesis effort was that of the French collective under the pseudonym *Nicolas Bourbaki*. Important members of the Bourbaki group were *André Weil* (1906–98) and *Jean Dieudonné* (1906–92). The group was born in 1935; the name goes back to a French general in the

Crimean War, somehow involved in a jest in the *École Normale*. Bourbaki's series *Éléments de mathématique*, in ten volumes, tries to present all important branches of mathematics in a unified axiomatic and logical framework. The influence of Bourbaki can be seen in terminology and notations: concepts such as *bijection*, *surjection*, the symbols  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  of the different number sets, the implication arrow  $\Rightarrow$  and the symbol  $\emptyset$  for an empty set are all *Bourbakisms*. – The Bourbaki group also initiated the *Séminaire Bourbaki*, active in Paris since 1948 as a forum for current and new mathematics.

## 17 On mathematics in Finland

Finland was – and still is, of course – in the periphery of the mathematical world. Mathematical innovation reached the northern latitudes with a delay of several decades. One searches almost in vain for Finnish names in mathematical encyclopaedias or general histories. But in certain subfields, Finnish mathematics has at times been quite advanced.

### 17.1 The beginnings

The beginnings of higher teaching and learning in Finland coincides with the founding of the *Turku Academy*, the predecessor of the University of Helsinki in 1640. The small faculty of the academy included a professor in *mathesis* which included mathematics and astronomy. This professorship was for more than 200 years the only office open for a mathematician.

The first professor of mathematics was *Simon Kexlerus* (Swedish, 1602–69). He had studied in Uppsala, Sweden. The level of Kexlerus' teaching and learning was fairly low. For instance, he put much effort in a defence of the Ptolemaic system against the Copernican one. The innovations of the 16th and early 17th century mathematics were unknown in Finland. Kexlerus' lectures dealt with the arithmetic of the sexagesimal number system and the elements of trigonometry and geometry. The infinitesimal calculus reached Finland in the middle of the 18th century, more than 50 years after it was discovered.

Two mathematicians in 18th century Turku, the most important city of Finland at that time, deserve to be mentioned. One is *Martin Johan Wallenius* (1731–73). In 1766 he published the complete solution of the generalized problem of Hippocrates on the lunes drawn on the sides of a regular  $n$ -gon. There are exactly five cases where the lunes can be squared in the manner of Hippocrates. Wallenius also initiated the teaching of the calculus in Finland.

The first Finnish mathematician to reach the international level was *Anders Lexell* (1740–84). He was born in Turku and he studied with Wallenius to reach the level of a university *docent*, but he then moved to St. Petersburg. He worked there as an assistant of the blind Euler. When Euler died in 1783, Lexell became his successor in the St. Petersburg Academy, but he, too, soon died. Lexell did important astronomical calculations and studied differential equations. He showed independently of Laplace that the celestial body discovered 1781 by *William Herschel* was the planet Uranus and not a comet as Herschel had supposed. Lexell obtained the professorship in Turku Academy, but he was all the time on leave in St. Petersburg. He did not leave any mathematical descendants in Finland.

A fire devastated the city of Turku in 1827. The university then moved to Helsinki, the new capital, and it was renamed as the *Imperial Alexander University*, to honour Czar Alexander I, who also had the title Grand Duke of Finland since 1809, after a war which made Finland a part of the Russian Empire. A gradual development towards the West European level of scholarship started. The first professor of mathematics in Helsinki, *Nathanaël af Schultén* (1794–1860) was still far from the frontline of science. But his writings deal with subjects of current interest: interpolation with trigonometric functions approximately at Fourier's time, the essence irrational numbers, continued fractions

and the foundations of differential calculus.

Schultén's successor in the chair of mathematics was *Lorenz Lindelöf* (1827–1908). Lindelöf's work in the calculus of variations was important, although he did not make any revolutionary breakthroughs. The textbook *Leçons de calcul des variations* which he published in Paris in 1861 was widely used in Europe. – Lindelöf was interested in politics and societal questions. He left his professorship in 1874 to become the director of the Central School Administration in Finland, a post he held until 1902. He was also a member of the estates-based parliament of Finland representing at times three of the four possible estates. While professor, he also served as the Rector of the university.

## 17.2 European connections opened

In 1877, Lorenz Lindelöf's successor assumed the office. He was a Swede, *Gösta Mittag-Leffler* (1846–1927), a student of Weierstrass. Mittag-Leffler's appointment created controversies because he did not speak Finnish, as was officially required at that time. Mittag-Leffler was an able function theorist and later, when he was active in Stockholm, an important figure as an organizer of international collaboration in mathematics. – Mittag-Leffler's home in the outskirts of Stockholm now houses the *Institut Mittag-Leffler*, an international mathematical research centre. Mittag-Leffler's considerable property which partly enabled the founding of the institute has its origins in Finland, from the inheritance of his Finnish wife, *Signe Lindfors*.

Mittag-Leffler's activities laid the foundations of the Finnish *function theoretic school*, which is the most important Finnish contribution to the general development of mathematics. During his four year stay in Helsinki, Mittag-Leffler had a few students, the most important of the *Hjal-*

*mar Mellin* (1854–1933). Mellin, too, for some time studied with Weierstrass in Berlin. His investigations of the gamma function and hypergeometric function led him to the integral transformation

$$F(z) = \int_0^{\infty} \Phi(x)x^{z-1}dx$$

and its inverse. This transformation is known as the *Mellin transformation*. – In his later years he became convinced of the incorrectness of Einstein’s theory of relativity: he published several articles in which refuted Einstein relying on ”common sense” arguments.

Mittag-Leffler’s successor in the professorship was *Edvard Rudolf Neovius* (1851–1917), a representative on the most important mathematical family in Finland, the *Neovius-Nevanlinnas*. Neovius studied in ETH, the technical university in Zurich where his teacher was *Hermann Amandus Schwarz* (1843–1921), a student and collaborator of Weierstrass; Schwarz is known for the *Lemma of Schwarz* in function theory and the *Schwarz inequality*, which, however, comes originally from Cauchy. Neovius studied projective geometry and the theory of *minimal surfaces*, originated by Riemann and Weierstrass. The question is about three-dimensional surfaces with a fixed boundary and minimal area. Neovius left the university for politics in 1900. He was a government member, but in the social turbulence in 1905 made his line which emphasized good relations with Russia unpopular, and he had to leave politics and the country. His wife was Danish, and he spent the rest of his life in Denmark.

Neovius played an important role in the consolidation of higher technical studies in Finland at the end on the 19th century. A *Technical high school* was founded in Helsinki in 1849 and in 1879 it was renamed as a *Polytechnic Institute*,

and in 1908 the institute was transformed to a Technical University. Mellin was the first mathematics professor in the new university.

### 17.3 Ernst Lindelöf and his students

The Finnish function theoretic school was essentially shaped by the long career as researcher and teacher of *Ernst Lindelöf* (1870–1946). Ernst Lindelöf, son of Lorenz Lindelöf, held the professorship from 1903 to 1938. His continental contacts were mainly in France, and like his father, he also published a monograph in Paris. The book was about the calculus of residues, a method originated by Cauchy to compute integrals of real functions using properties of line integrals of complex functions and the Cauchy integral theorem. In his research work on function theory, Lindelöf considered entire functions or analytic functions without poles or points where the function assumes the value infinity, and conformal mapping. His studies of entire functions anticipated the emergence of the value distribution theory of analytic functions. Lindelöf's collaboration with *Edvard Phragmén* (Swedish, 1863–1937) resulted in the well-known *Phragmén–Lindelöf theorem* on the behaviour of an analytic function in a domain of the form of a sector reaching to the point of infinity.

Lindelöf's interests extended to other fields of mathematics, too. He contributed to the existence proofs of solutions of differential equations, proved the *Lindelöf cover theorem* related to the Heine–Borel theorem in point set topology. The well-known and still unproved *Lindelöf hypothesis* is related to the Riemann hypothesis. In his later years, Lindelöf concentrated in teaching. He wrote a series of text-books in Finnish (Lindelöf was a Swedish-speaking Finn, and his books were translated before publication) for the university curriculum, rounding up the four volume *Differentiaali- ja*

*integraalilasku ja sen sovellutukset* (Differential and integral calculus and its applications) in a treatise of the calculus of variations, to honour the work of his father.

Important colleagues and students of Lindelöf, mainly active in function theory, were *Severin Johansson* (1879–1929), *Felix Iversen* (1887–1973), *Pekka Myrberg* (1892–1976) and the brothers *Rolf Nevanlinna* (1895–1980) and *Frithiof Nevanlinna* (1894–1977).

In 1918 a new, Swedish-speaking university *Åbo Akademi* was established in Turku as well as a new Finnish university. Severin Johansson was its first mathematics professor in Åbo Akademi. He initiated the study of automorphic functions in Finland. (In Ernst Lindelöf's opinion, automorphic functions were too difficult, and he did not want to deal with them.) Felix Iversen's speciality were the singularities of meromorphic functions. He was also known as a philanthropist and pacifist and for his four decades as the secretary of the State Matriculation Examination Board. Pekka Juhana Myrberg's work on automorphic functions was internationally recognized. He served as professor in the Technical University and the University of Helsinki, and was finally the chancellor of the university. – Myrberg's participation in a competition organized by the Jablonowski Society in Germany. The problem was to investigate functions satisfying an algebraic addition theorem (such as  $f(x + y) = f(x) + f(y)$ ) and the set time was four years. Myrberg only learned of the competition when the last year was running, wrote a paper and won the competition. Later, when administrative duties took his time and he could not concentrate on research, he wrote a series of papers on iteration of rational functions. In this he proved to be ahead of his times. The problems he dealt with became important in the study of fractals.

All students and colleagues of Lindelöf were not function

theorists. *Jarl Lindeberg* (1876–1932) was a versatile and independent researcher. He worked in potential theory, calculus of variations, probability and mathematical statistics. His proof of the central limit theorem in probability (the sum of stochastic variables with almost any distribution approaches the normal distribution, when the number of terms in the sum increases) is famous. *Nils Pipping* (1890–1982), professor in Åbo Akademi, worked in number theory and *Kalle Väisälä* (1893–1968), first mathematics professor in the Finnish University of Turku and later professor in the Technical University in algebra and algebraic number theory. Turku evolved into a centre of number theory and algebra in Finland. An important representative of the number theoretic school of Turku is *Kustaa Inkeri* (1908–1997). Inkeri's speciality was Fermat's Last Theorem. Later, the theory of automata and coding theory have had important representatives in Turku, such as *Arto Salomaa* (1934 –). – Kalle Väisälä's influence on high school mathematics teaching in Finland is to be noted. He voluntarily started teaching at high school level to be able to write a series of text-books, which were widely used in the 1950's and 1960's.

#### **17.4 Rolf Nevanlinna and value distribution theory**

There are exceptionally many mathematicians in the family Neovius-Nevanlinna. The grandfather of Rolf and Frithiof Nevanlinna, major general *Edvard Engelbrekt Neovius* (1823–88) was a teacher of mathematics in the Hamina Cadet School, their father *Otto Nevanlinna* (1867–1927) the senior mathematics teacher in a State Normal High School, their uncle Edvard Neovius a mathematics professor and another uncle *Lars Neovius-Nevanlinna* (1850–1916) a mathematics teacher and author of widely used text-books. The son of Frithiof Nevanlinna *Veikko Nevanlinna* (1920–) was

a professor in the University of Jyväskylä, a nephew of Rolf and Frithiof, *Heikki Haahti* (1929–) was a professor in the University of Oulu and *Olavi Nevanlinna* (1948–), professor in the Technical university, is Frithiof's grandson.

Rolf Nevanlinna is often considered to be the most important Finnish mathematician. His fame rests primarily on the general theory of the distribution of values of meromorphic functions, discovered in 1925. (A function  $f(z)$  of a complex argument  $z$  is *analytic* or *holomorphic*, if it has a derivative at every point of its domain. Then it has a Taylor series development at every point. If the domain is the complex plane, the function is *entire*. The function is *meromorphic*, if infinity is accepted to the range but the function still has a Laurent series development at every point of the form

$$f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^k.$$

The singularities of the function then are *poles* of order at most  $n$ .) The starting point of value distribution theory was the result by *Emile Picard* (French, 1856–1941) from 1879, stating that an entire function assumes all complex values with one possible exception. That the exception can exist is shown by  $e^z$ , which never takes the value 0.

Picard's result had been improved by Borel, but the results published by Nevanlinna in the mid-1920's concerned the much more general class of meromorphic functions and were far-reaching. They are known as the first and second main theorems of Nevanlinna theory. Their essential content is that a meromorphic function assumes every complex value approximately as often as any other value. But should some value be underrepresented among the values of the function, then the function comes close to this value more often than the "ordinary" values. In 1936 Nevanlinna

published his theory in a monograph form as *Eindeutige analytische Funktionen*. – Frithiof Nevanlinna also contributed to the discovery of the Nevanlinna theory. It has remained an important subfield in complex analysis. Generalizations to several directions have been made.

Rolf Nevanlinna was active in several fields. He published on Riemann surfaces, functional analysis and the axioms of geometry. In the years of the second world war, Nevanlinna served as Rector of the University of Helsinki. He had a German orientation, partly influenced by the fact that his mother was German. After the war, it was impossible for him to hold a high position in the university. In 1948, the Academy of Finland was created as an institution supporting a dozen important scientists and artists. Nevanlinna was the first (and only) mathematician in the Academy. He also served long as a half-time professor in the University of Zurich. A tradition of Finnish-Swiss *Rolf Nevanlinna colloquiums* arose from Nevanlinna's Swiss connections. Frithiof Nevanlinna worked in leading positions in insurance business, but served as professor in Helsinki for some years in the 1950's.

## 17.5 Function theory after Nevanlinna

The founding of the Academy of Finland in the turbulent time after the war was very much due to the energy of a young post graduate student *Leo Sario* (1916–). Sario, a student of Nevanlinna, later became a professor of the University of California and a known researcher in Riemann surfaces. Rolf Nevanlinna, on the other hand, was active in the mid 1960's when the old Academy was discontinued and a central science administration bearing the same name was established.

Rolf Nevanlinna's first doctoral student, and the most im-

portant one, was *Lars V. Ahlfors* (1907–1996), a versatile complex analyst and the only Finn ever to receive the Fields Medal. Ahlfors worked mostly abroad. After the war he went to Zurich and from 1946 he was professor in Harvard University. Ahlfors created a small sensation when he, at a young age, solved an open problem presented much earlier by *Arnaud Denjoy* (French, 1884–1974). Later Ahlfors developed a more geometric version of Nevanlinna’s theory, the theory of *covering surfaces*, and he was the first to understand and appreciate the classification theory of Riemann surfaces discovered by *Oswald Teichmüller* (German 1913–43?). Teichmüller had used *quasiconformal mappings*, and Ahlfors established these functions as an important tool in complex analysis. Quasiconformal mappings, which became an important object of study in Finland resemble conformal mappings. Basically, a conformal mapping maps an infinitesimal circle on a circle, but a quasiconformal mapping takes an infinitesimal circle onto an ellipse, but one with a bounded ratio of the major and minor axes.

Among the many mathematicians who have worked with quasiconformal mappings in Finland, *Olli Lehto* (1925–) is one of the most important. Lehto was a student of Nevanlinna. Together with *K.I. Virtanen* (1921–2006) he published in 1965 the monograph *Quasikonforme Abbildungen*, which was the main reference of quasiconformal mappings for a long time. Lehto also served as the Rector and Chancellor of the University of Helsinki. He has earned credits in the international organizations of mathematics and written biographies of Rolf Nevanlinna and the Lindelöfs. In Helsinki, the study of quasiconformal mappings and their generalizations to dimensions higher than two – one important figure in this work has been *Jussi Väisälä* (1935–), son of Kalle Väisälä – has been a natural continuation of the Finnish function theoretic school.

## 18 On calculating devices and computers

When mathematics is applied for some practical purpose, almost always some calculations have to be performed. A consequence of the development of calculating devices has been that theoretical advances in mathematics often follow from calculations and use of computers.

### 18.1 Mechanical devices

As devices for storing numbers – in addition to graphic signs – for instance knots in strings or carvings in duplicate sticks have been used. Widely spread conventions to express numbers with certain positions of fingers existed at least in the times of classical antiquity. Numbers smaller than 100 were shown with left and larger ones with right hand.

The oldest mechanical calculation aid is the *abacus*. Abacuses exist in many forms in different cultures, sometimes with wires and beads, sometimes in the form of a board with grooves and small stones, *calculi*, showing the amounts of various units. (Thus the etymology of *calculus* goes back to the latin word for limestone.) Among the early calculating aids are *Napier's rods* or *Napier's bones* invented by John Napier around 1600. With them, one could perform gelosia-type multiplications.

Essentially more advanced than the abacus is a mechanical device which automatically takes care of the operation of carrying in additions. The idea of such a device was pre-

sented in 1623 by the astronomer *Wilhelm Schickard* (1592–1635), a friend of Kepler's, and the Jesuit *Johann Ciermans* (1602–1648) in 1640, but the first working machine was built by *Blaise Pascal* in 1642, to assist his father's calculation as a tax official. Seven years later Pascal was given a royal privilege to manufacture calculating machines. The first machines capable of multiplication were built by *Samuel Morland* (English, 1625–95) and Leibniz in the 1670's. A working machine which could do all the four basic arithmetical operations was not built until 1820 by *Charles Thomas* (French, 1785–1870). The pinwheel calculator, which was the common type of calculator before the age of electronics, was developed in the 1870's by *Frank Baldwin* (American, 1838–1925) and *Willgodt Odhner* (Swedish, 1845–1905). In these machines, multiplication was performed as repeated addition and division as repeated subtraction. The first machine directly utilizing a multiplication table was constructed by *Léon Bollée* (French, 1870–1913) in 1887.

The invention of logarithms around 1600 was of course a remarkable step in facilitating numerical calculations. That logarithms could be utilized also mechanically was first observed by *Edmund Gunter* (English, 1581–1626). In 1620 he constructed the logarithmic scale. With it and a pair of compasses, multiplications could be performed. *William Oughtred* (English, 1574–1660) discovered in 1622 that two such scales could be combined in a *slide rule*. Oughtred and his student *Richard Delamain* (1600–44) also developed the circular slide rule. Oughtred's slide rule still lacked the cursor, the idea of which was outlined by Isaac Newton in 1675 although the device did not become common until a century later. The standard modern type of slide rule was essentially presented by the French officer *Amédée Mannheim* (1831–1906) around 1850.

## 18.2 From Charles Babbage to computers

*Charles Babbage* (English, 1792–1871) holds a special position in the history of artificial computing. Babbage started as a mathematician. He was professor in Cambridge and an active member of the *Analytical Society*, which tried to substitute the outdated methods of Newton by the Analysis of Leibniz and Euler. Babbage was irritated by the frequent errors in mathematical tables, and from 1812 he started to think of methods by which one could carry out mechanical calculations by machines. Babbage gave up his university position and devoted all his energies to calculating machines. He used all his property in the process but in 1823 he got a substantial grant from the government for constructing a *difference engine*. The machine was supposed to calculate values for mathematical tables at an accuracy of 26 significant digits, and take into account the sixth difference. After ten year's work the machine was still uncompleted. The government withdrew its support, but Babbage continued with his own means. His plan had grown considerably: from 1834 on Babbage worked on the *analytical machine*. The plan called for a mechanical memory capable of storing a thousand 50 digit numbers and a processor which could operate following programs which included branching commands. Babbage had as his assistant *Lady Ada Lovelace* (1815–52), daughter of the poet *Byron*. She wrote programs for the machine. A punched card mechanism was to feed the programs for the machine. The fine mechanics needed for the extremely complicated actions was not good enough, and only parts of the machine could be physically completed. – Difference engines in the lines of Babbage's idea were built in the 1850's for instance in Sweden. A working Babbage's difference engine was constructed from his drawings in the Science Museum in London in 1985 – 1991.

A pioneer in a sense of data processing technology was the weaving machine constructed by *Joseph Jacquard* (French, 1752–1834) in 1805. Its complicated directives were coded on punched sheets of cardboard and automatically fed to the machine. *Hermann Hollerith* (American, 1860–1929) realized the significance of the punched card for saving information around 1885. Hollerith's card machines were used in a grand scale in the United States census in 1890. Hollerith's firm became in 1911 IBM, later the leading manufacturer of computers. The binary coding of numbers in the cards was introduced in 1934 by *Konrad Zuse* (German, 1910–95) in 1934.

Mechanical methods for determining the area enclosed by a given curve started to develop around 1810. *Jacob Amsler* (Swiss, 1823-1912) constructed in the *planimeter* in 1854. It was a commercial success. Similar but more complicated instruments, *harmonic analyzers* were made from 1870's to determine mechanically the Fourier coefficients of a function represented by its graph.

First significant steps towards the modern computer were the electromagnetic analogy computers constructed by *Vannevar Bush* (1890–1974) in Massachusetts Institute of Technology from 1925. Closer to Babbage's ideas was the electromagnetic *MARK I* started by *Howard Aiken* (1900–73) in Harvard in 1939. Aiken was supported by IBM. The machine was completed in 1944 and it was used by the US Navy. Before the Second World War Zuse had built machines using electromagnetic relays and binary coding of numbers. His plans to replace the relays by electronic vacuum tubes could not be realized because of the war.

The first strictly electronic computer was the *ENIAC* (*Electronic Numerical Integrator and Computer*), a joint effort by the university of Pennsylvania and the US Army. The machine was completed in 1945 and it was used for com-

puting artillery tables. The machine was planned by *John Mauchly* (1907–80) and *J. Presper Eckert* (1919–95). It contained 18 000 electron tubes and it weighed 30 tons. A couple years earlier a computer using electron tubes, the *Colossus* was built in England. It was used exclusively in deciphering German coded messages.

In 1945 the main principle of a modern computer was presented by John von Neumann: the directions for the machine can be stored in the memory of the machine in the same way as the numbers or other data the machine is supposed to process. First machines built with von Neumann's principle were the *EDVAC* and *EDSAC*; they were constructed in 1947 in Princeton and Cambridge (Mass.). The first commercially available computer was the *UNIVAC*, built by the planners of the ENIAC in 1951. From 1954 on, transistors came to use. The size of computers diminished and their reliability increased.

### 18.3 Mathematics and computers

Already before the modern computation technologies became reality, the English mathematician *Alan M. Turing* (1912–54) presented a generic model of a universal computer which in principle can solve all problems which can be solved by computations. The *Turing machine* consists of an infinite strip divided in squares. Each square can hold a number and the strip can be moved forwards or backwards according to the number in the square which is observed by the processor. Turing's ideal machine and the results of the American logician *Emil Post* (1897–1954) gave rise to the mathematical theory of computation.

During World War 2, Turing's work was instrumental in British attempts to break German codes, especially the code produced by the so called *Enigma* machine. Immediately

after the war Turing held a central position in the English ACE computer project, which had in many respects more advanced than the American ones of the period.

Of the established branches of mathematics, the method and practice of numerical analysis has of course gained much from the use of computers. But the immensely greater possibilities to compute or repeat defined operations almost without limit have been useful on many different branches of mathematics, up to number theory. A considerable attention was created by the computer solution of a century old topological or graph theoretical problem, the *four colour problem* in 1977: a program checked a great number of cases which proved that it indeed always is possible to use only four colours to colour a map showing simply connected regions in such a way that adjacent regions always have different colours.

Computers give possibilities to experiment and visualize. There are opinions that the role of traditional proof in mathematics is diminishing when a geometrical "truth" can be directly seen in a visualization or when overwhelming numerical support can be given to a hypothesis. Probably it is still premature draw final conclusions on the meaning of computers to the methods and ways in which new mathematical theory is created.

## 19 The mathematical community

Mathematics is an autonomous theory but also an endeavour of the human community. Cooperation between mathematicians is certainly much older than its organized forms. For the development of mathematics, the dissemination and availability of information are paramount. Before the era of electronic information networks, information could be moved either on paper or, say, by moving people with information to a common meeting place like a congress. Some forms of mathematical cooperation are not overly serious. One example is the *Erdős number*. *Pál Erdős* (1913–96), a versatile Hungarian mathematician wrote an exceptionally large number of articles together with other mathematicians. By definition, the Erdős number of Erdős is zero, and a mathematician has Erdős number  $n$ , if he has published an article together with a mathematician with Erdős number  $n - 1$ .

### 19.1 Mathematical publications

The standard way to publish a mathematical result is an article in a mathematical journal. Publications specializing in mathematics are a relatively young phenomenon, the first one might be the *Journal de l'École Polytechnique*, appearing from 1794. In the 18th century, the mathematics was published in general scientific journal like the *Acta Eruditorum* and in the publication series of various scientific academies. Among the oldest scientific academies rank the *Accademia dei Lincei* (the lynx is considered to be a very

sharp-sighted animal), established in 1603 in Rome, The *Royal Society* in England since 1662 and the *Académie des Sciences* in France since 1666.

The first privately established mathematical journal was the *Annales de Mathématiques Pures et Appliquées*, founded in 1810 by Joseph-Diaz Gergonne. The Annals of Gergonne did not live long, but their German counterpart, the *Journal für die reine und angewandte Mathematik* established by August Crelle (1780–1855) in 1826 is still active. Joseph Liouville started in 1836 a new French journal with almost identical name *Journal de Mathématiques Pures et Appliquées*; a large part of the advances of mathematics in the 19th century were recorded on the pages of the Journals of Crelle and Liouville. At the turn of the 19th and 20th centuries, the journal *Acta Mathematica* which was edited by Gösta Mittag-Leffler from 1882 assumed an important position in mathematical publishing. Mittag-Leffler, a man of many contacts, stood on a neutral ground when relations between the two greatest mathematical nations, Germany and France, became hostile.

The first scientific academy in Finland is the *Societas Scientiarum Fennica*, The Finnish Society of Sciences and Letters. It was established in 1838 and much of the mathematics written in Finland in the 19th century has been published by the Society. The first secretary of the society was af Schulten, and Lorenz and Ernst Lindelöf both had the office for decades. In 1908, as a consequence of language disputes, a rival Finnish-speaking academy, *Academia Scientiarum Fennica* was established. From 1940, its publication *Annales Academiæ Scientiarum Fennicæ Series A. I. Mathematica* devoted to mathematics and physics, then for mathematics only, became the most important channel for mathematics publication in Finland, under the editorships of P.J. Myrberg and Olli Lehto.

Greatly increased amount of publishing in mathematics and its distribution to hundreds of journals made review journals indispensable. The first such publication was the *Jahrbuch über die Fortschritte der Mathematik*, Annals of the Advances in Mathematics started in Germany in from 1872 to 1942. It was followed and replaced by *Zentralblatt für Mathematik* in the 1930's. The American rival, *Mathematical Reviews* was established in 1940. In the Soviet Union and Russia, a similar publication is the *Referativnyi Zhurnal Matematyki* which was started in the 1930's.

## 19.2 On mathematical organizations

Many mathematical journals are published by mathematical societies. The oldest societies devoted to mathematics might be the Mathematical Societies of Hamburg (established in 1690) and Amsterdam (1776). The Moscow Mathematical Society was founded in 1860 and the London Mathematical Society in 1865. The Finnish Mathematical society, founded by Lorenz Lindelöf, is only three years younger. Probably the largest of the present day scientific mathematical societies, The *American Mathematical Society* is from 1888 and the German *Deutsche Mathematikervereinigung* from 1890.

A very visible form of mathematicians' international collaboration is the quadrennial *ICM*, *International Congress of Mathematicians*. The first congress was held in Zurich in 1897. Gösta Mittag-Leffler was instrumental in its preparations. The number of participants was 200. The next congresses took place in Paris (1900), Heidelberg (1904), Rome (1908) and Cambridge, England (1912). From the very start, the congress program has consisted on general lectures exposing current developments and more special sectional lectures.

The ICM of 1924 was in Toronto. Its chairman, the Cana-

dian *J. C. Fields* (1863–1932) proposed that the economic surplus of the congress should be transformed to a prize fund for giving a special prize to two young mathematicians whose achievements. Thus the *Fields medal*, the closest equivalent to the Nobel Prize in mathematics, was born. (In his will, Fields gave more money to the fund.) The first Fields Medals were given in the Oslo ICM in 1936 to Lars Ahlfors and *Jesse Douglas* (American, 1897–1965), whose work was on the calculus of variations. In monetary terms, the Fields prize is extremely modest compared to the Nobel Prizes. – That mathematics is not one of the five areas of human action which Alfred Nobel considered worthy of rewarding by his prizes is a fact often discussed. There have been suggestions that a personal animosity between Nobel and Mittag-Leffler might be the reason. The only persons mentioned in this book who have received a Nobel Prize are Albert Einstein and Bertrand Russell, who was awarded the prize in literature in 1950. Of course, among the recipients of the physics prize there have been several mathematically outstanding persons as well as among the recipients of the Economy prize which was not included in Nobel's original arrangement.

The ICM's long operated without any permanent international organization to back them. Efforts made in the 1920's to create an international mathematical organization were crushed by the political events in the 1930's. After the Second World War the *International Mathematical Union, IMU* was established. Rolf Nevanlinna served as the President of the Union in 1959–62 and Olli Lehto as its Secretary in 1979–90. The IMU now takes responsibility of the program of the ICM's as well as of the distributing the Fields Medals. The year 2000 was celebrated as the World Mathematical Year on the initiative of the IMU. A subcommittee of the IMU is the *International Commission of Mathemati-*

*cal Instruction, ICMI*. The Commission was set up already before the First World War. The ICMI also arranges a large quadrennial general meeting, the *ICME* or *International Congress of Mathematical Education*.

The number mathematical congresses, symposiums, colloquiums etc. of different kinds is large. There are meetings based on the geographical distribution of the participants. The quadrennial *Scandinavian Congresses of Mathematicians* started in 1909 in Stockholm, on the initiative of Mittag-Leffler. Mathematical organization on a European basis is much younger. The *European Mathematical Society* was established in 1990 and the first *European Congress of Mathematics* was held in Budapest in 1992.

### 19.3 Mathematics and women

Almost all mathematicians who have left their name in history are men. The main reason for this is probably just the fact that careers in learning have not been open to women in the past. The sex distribution is quite similar in most fields of science.

The female mathematicians already mentioned in this book are the mathematicians Hypatia, Maria Agnesi and Emmy Noether and the pioneer of programming, Ada Lovelace. Some other female mathematicians can claim a position at least in the marginal of the history of mathematics.

*Sophie Germain* (French, 1776–1831) entered in a correspondence with Gauss under a masculine assumed name *M. Le Blanc*. She contributed to the proof of Fermat's last theorem in some special cases.

*Sofia (Sonja) Kovalevskaya* (Russian, 1850–91) was a private student of Weierstrass. She obtained important results on the existence of solutions to partial differential equations and in mechanics. Mittag-Leffler arranged a professorship

for Kovalevskaya in Stockholm in 1883. Kovalevskaya thus seems to be the first female mathematician to hold a university position. In most countries, the percentage of women in faculty positions still is rather low.

## 20 On the philosophy of mathematics

The philosophical conceptions about mathematics prevailing around the turn of the 19th and 20th centuries can be roughly divided in three schools, the *logistic*, *intuitionist* and *formalist* school.

The chief thesis of the logistic school can be crystallized as the thought that mathematics is a branch of logic. There are no particular first axioms from which mathematics starts. Pioneers of the school were Dedekind, Frege and Peano, but its culmination was the monumental *Principia Mathematica* by Whitehead and Russell. The aim of the *Principia* is to derive the system of natural numbers and thence all mathematics based on it from logical primitives, which themselves can be considered to be sensible descriptions of the real world. The predicate calculus and the theory of classes and relations are derived from the primitives. The paradoxes of set theory are fended off by the theory of types. There is not a general consensus on the success of *Principia's* program.

The roots of intuitionism go back to the views of Kronecker, who for instance rejected irrational numbers. Poincaré and several other French mathematicians favoured the intuitionist view.

The programme of intuitionism was formulated by Brouwer in 1912. According to him, the primary source of mathematics is the human mind's perception of time and the order associated with it. This leads to the system of natural numbers. So man has a natural intuition of numbers and this

intuition is independent of language or experiences gained from outside. Logic is a generalization of mathematical relations among finite sets, and so it is not more primitive than mathematics.

The basic principle of action of intuitionism is *constructivism*. Mathematical results have to be derived by finite, constructivist methods. One of the most common phrases in mathematical literature is "there exists an  $x$  such that ...". By the intuitionist's conception, the existence of an object cannot be proved by showing that the assumption of its non-existence implies a contradiction. The object has to be constructed by finite operations from objects whose existence is known, from natural numbers in the final analysis. Induction for instance is an insufficient basis for an existence proof. Statements concerning infinite collections may become problematic. We are not allowed to say that the statement "in the decimal expansion of  $\pi$  there is the sequence 123456789" is either true or false, at least before such a sequence is actually found. The Euclidean proof for the infinity of primes is not valid, but the definition of a prime is good, because the property of being prime can be tested with finitely many operations.

The most important later developer of intuitionist mathematics was *Arend Heyting* (Dutch, 1898–1990). In 1930 he published a system of symbolic logic conforming to the intuitionist system.

The formalist school was founded by Hilbert. His axiomatization of geometry (1899) served as a model for axiomatic systems whose working does not depend on the concrete interpretation of the concepts involved. The paradoxes of set theory and the threat of a radically narrowed scope of mathematics inherent in the intuitionist conception inspired Hilbert's 1920's view of mathematics as a purely formal system which has no "content" in itself. The essential feature

of the system is its *consistency*. The system is not allowed to have a statement or formula like " $P$  and not  $P$ ". Hilbert tried to find a proof for the consistency of mathematics or at least as large as possible part of mathematics. There were some partial successes such as the first order predicate calculus, and Hilbert and his school strived to prove the consistency of the axiom systems of set theory and arithmetic. But in 1931 *Kurt Gödel* proved that every reasonably complicated mathematical system such as the system of natural numbers based on Peano's axioms contains statements which are improvable by methods belonging to the system. And one such statement happens to be the consistency of the system. Gödel's proof was done by assigning numbers to arithmetic concepts, operations and theorems which made statements about these, such as "this statement is provable" themselves material of arithmetic.

Still further from mathematics such as it appears in the daily work of mathematicians go discussions of questions such as whether mathematics is a product of man or whether it has an existence independent of man and his discoveries. The latter view is known as *Platonism*.

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