

Mathematics in Operations Research

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Abstract

An overview of the use of mathematics in some established operations research methods is given. The implications of the needs of operations research mathematics on officer education is discussed.

1 Introduction

In a sense, serious and good science can be characterized by a certain dualism. At the same time, it aims at solving some of the very concrete problems facing mankind or some subset of it, and at the same time it tries to develop a general understanding – that is theoretical knowledge – of the phenomena it deals with. Operations research is not an exception in this respect. In considering the relationship between mathematics and operations research, this division is to be noticed.

There are situations where operations research solutions to practical problems are just clever insights, without any particular mathematical connotations. The anecdotal elevator problem described e.g. in [5], where customer dissatisfaction with long waiting times for the elevators in a tall building was met not by further elaborating the calling algorithms of the elevator system but by putting up mirrors beside the elevator doors to give people a sight with which they liked to spend the actually quite small waiting time, is an example of this kind of problem solution.

However, it is usually taken for granted that operations research is a mathematical discipline, or – to quote Thomas L. Saaty – at least a "legitimizing of quantitative methods in Everyman's everyday operations" [3]. This is emphasized in the widely used parallel name operational analysis. The use of the Greek word *analysis*, 'dissolving', in the sense of finding the pieces of which to build something, for instance an operation, has a long tradition in mathematics. There it has evolved from being a method of finding a geometrical proof by tracing the needed steps in the reverse order into a general concept covering essentially all mathematics that deals with continuous phenomena.

Operational analysis comes closer to the original meaning of the word. From a mathematician's point of view, there is much in the mathematics of opera-

tions research that would not be labelled analysis. In this article, we review on a general level some of the mathematical ideas present in operations research methods and reflect the need to educate potential users of operations research methods in mathematics. We restrict ourselves to rather simple and not so technical aspects of mathematical associations of operations research methods. Our particular concern is the education of military personnel. So our focus is in the mathematics of military operations research.

2 Mathematics and its applications: mutual interaction

When looking back in history, we can distinguish mathematics as a recognizable element in human culture for at least four millennia. There are two fundamental strains in this history. One can be described as that of pure mathematics, going back to Greek geometry and arithmetic, the other as applied mathematics, going back to early civilizations, all of which all needed numerical tools to handle activities binding societies together, e.g. commerce and taxation. (It is amusing to notice that the Greek expression of practical computing, *logistika*, now reappears as the art and science of transportation, storage and related matters.) In a way, the computing side has its ancestors in the civilizations which in the history of mathematics are bundled up as Babylonian. Modern usage and machinery again distance the methods and practice of computing from mathematics proper, despite the fact that basic arithmetical skills are taught as "mathematics" in primary schools.

The different strains of mathematics started to merge together in the 17th century when mathematical analysis emerged as the major tool in the study of the the dynamic phenomena in the Newtonian world and universe. The development of mathematical analysis took place in parallel with the development of physics. This does not rule out the fact that there already was a lot of off-the-shelf mathematics available. A glorious example is the theory of conics, developed largely by Apollonius of Perga in 3rd century B.C, which was ready for use for Kepler and Newton in their model of central motion and gravity.

A clear conception of a division of mathematics proper into "pure" and "applied" sections is to be noticed already in early 19th century, as is evident for instance in the name *Journal für die reine und angewandte Mathematik*. The application area of mathematics was understood to be physics. A relatively much younger development is the application of mathematics in various somewhat less exact sciences, from cryptography and economics to psychology and biology, to mention just a few. When dealing with operations research we mostly encounter applied mathematics in this wider sense.

One cannot exactly pinpoint the birthdate of operations research. A widely shared opinion favours the publication of F.W. Lanchester's attrition equations [2] as the crucial event in this respect. It is interesting to note that the battle Lanchester describes is a dynamic phenomenon and the tools he uses are from

classical mathematical analysis, i.e. coupled differential equations.

Operations research probably never reached or reaches such a crucial status in the shaping of our understanding of the world as physics enjoyed in the centuries after Newton. So we cannot expect that it would very much contribute to opening quite new domains to mathematical research. Rather, the mathematics used in operations research is mostly existing mathematics, tailored to the needs of problems at hand.

3 Where does mathematics enter in operation research?

In the standard description of the operational analyst handling an operational analytic job a number of successive steps are included. The analyst receives the problem somewhere, probably from a superior, he recognizes the variables involved and builds a model of the situation, gathers data to tune the parameters of the model, he obtains a solution, either one in closed form or obtained by numerical simulation, possibly stochastic, validates the solution by a comparison to hard facts or (as is often the case in military problems) by a comparison to expert opinion, and tries to get his results implemented, say by the superior who in the first hand was asking for advice.

In this process, clearly the modelling and solution phases, possibly, involve mathematics. The word model has a wide scope. There are cases where a model is a physical thing: we may model a network with edges of variable size by actually binding together pieces of string and stretching the configuration to obtain the shortest path. But we can also model a network by a typically mathematical object, say an adjacency matrix where the matrix entries may stand for the distance of the vertices. Or we can model a process by describing the velocities in which the variables change when interacting. Thus a model can be a system of differential equations, like in classical physics.

Thus, although the most important thing in modelling is a clear understanding of the basics of the situation, that is not enough. One should know mathematics enough to know what kind of tools are available and one should have the technical competence to write down the equations, inequalities, differential equations, matrices, or whatever the problem calls for.

If the model is mathematical, then certainly the solution phase involves mathematics. There are alternatives. A mathematician's solution is a solution in closed form: the outcome may be a function of time and parameters of the problem describing the phenomenon at hand and giving a possibility to choose such values for the parameters as yield the optimal results for the one who has ordered the study.

Given the variety of problems, a closed solution is a rare exception. The huge computational capacities available now make numerical solutions and solutions based on stochastic simulations more attractive. Just having a computer does not

suffice. The numerical methods have their own limitations which may play a role in the solution process. Thus, an operational analyst needs some knowledge of the methods and ramifications of numerical analysis. Another story is stochastic simulation. When uncertainties are imitated by pseudo-random processes, basic understanding of probability is needed. And, as the simulation results vary from run to run, one has to be able to interpret them statistically.

4 Mathematical ideas in certain operations research areas

We give a superficial and brief survey of some well-established subfields of operations research and their mathematical tools.

4.1 Optimization

In a sense, most questions in operations research boil down to an optimization question. This is most evident in the practical side of the science. Decisions are made expecting a good value, and research aiding the decision maker should point the way to maximal gains. What is maximal gain is of course often a complex problem.

4.1.1 Classical optimization methods

In basic differential calculus courses we are taught how to handle simple optimization tasks. In these tasks, there is a continuous function of one variable, and conditions for local or global extrema are derived in terms of the derivatives of the function. The main idea is that something which has reached its maximum or minimum is not growing or diminishing just at the extremal point.

A problem originating within operations research rarely is of the simple kind described above. Rather, the number of variables is larger, there are constraints closing certain combinations out, the functions – or the variables – are not continuous. Thus the spectrum of optimization techniques needed in operations research is varied and includes many *ad hoc* -type solutions.

From the classical point of view, the partial derivatives – or gradient vector – still provide a simple indicator of the growth of the function to be optimized, and their vanishing is a hint of a possible optimality. The role of boundary conditions becomes crucial and makes the use of classical methods much more complex. There are methods within classical analysis – such as the Lagrange multipliers – that can take care of the boundary conditions. But these techniques are not considered to belong in the mainstream of operational research optimization methods.

4.1.2 Linear optimization

Most textbooks on operations research contain a lot of material on linear optimization. In its basic formulation, the linear optimization problem calls for the

extremum of a real linear function

$$f(x_1, \dots, x_n) = \sum_{k=1}^n a_k x_k \quad (1)$$

of a certain number of real variables. The domain of the function is defined by a number of linear constraints on the variables, i.e. inequalities

$$\sum_{k=1}^n c_{ik} x_k \leq b_i, \quad i = 1, \dots, m. \quad (2)$$

The task is simple and not interesting, if we rely on the well-established algorithms for the solution. There is no shortage of computer implementations of these algorithms.

Although the task seems simple, a number of very fundamental mathematical ideas are involved. One is the concept of an n -dimensional linear space, where each of the constraints (2) defines a half-space. Then there is the idea of convexity, which is preserved in the operation of set intersection and thus characterizes the feasible domain G or the part of the n -dimensional space where the problem is relevant. This domain is a n -dimensional polyhedron bounded by $(n - 1)$ -dimensional hyperplanes and their lower-dimensional intersections. The value of f clearly is constant in any hyperplane the normal of which can be read from the coefficients of the linear expression of f and is the constant gradient vector of f . It is intuitively clear, then, that the extremum of f , if it exists, is reached when we move in the direction of the gradient as far as is possible without losing contact to the convex feasible domain, and the extremum thus can be found on a single vertex point or a some larger part of the border of the feasible domain.

What was said above can easily be visualized in two- or three-dimensional space. The computational methods utilized in problems in which a much larger number of variables is involved are then suggested by the low-dimensional analogy. In particular, the celebrated simplex method which is the backbone of the computational arsenal available for linear optimization problems, is readily suggested by the desire of reaching the vertex which is "uppermost" in terms of the gradient of f , by following a as steeply as possible "rising" path along the edges of the polyhedron G . The not very transparent computational procedures of the simplex method, which occupy a lot of space in the text-books, are a translation of these ideas into the language of the closely intertwined mathematical disciplines analytic geometry and linear algebra.

It is not necessary to have this geometric-topological view when formulating a linear optimization problem, say, within logistics. Then it is probably enough to recognize the variables of the problem and have a sufficient command of the computational tools needed. The value of understanding the inner workings of mathematics may come to the fore when a problematic computational situation

due to an error arises. Our chances to restart a broken engine are certainly better if we know how the engine works. Our chances to recognize an error and recover from it in a computational task increase significantly if we understand the principles involved.

4.1.3 Other optimization techniques

The geometric ideas of linear programming can be found in certain other optimization contexts where the aim is to maximize or minimize a function of one dependent variable. These include quadratic and other non-linear problems and problems involving discrete variables, the so called integer programming problems. The ingenious methods used in the solutions do not bring any basically new point of view, rather they are clever approaches which can be said to be problem-oriented.

In a sense, a simple but deep and new idea of geometrical nature is basic to Richard Bellman's well-known dynamic or multi-stage optimization: given a point B on the shortest path Γ from A to C , the shortest path from B to C must go along Γ . The idea enables one to solve an optimization problem by breaking it down to parts which as themselves are more likely to find a solution than the global problem originally posed.

4.1.4 Multi-criteria optimization

The optimization problems encountered in practice are rarely of the type involving just one scalar-valued function. Probably most of the problems in which one might want to know the optimal solution are more complex. Often they still reduce to optimizing a vector-valued function. The components of the problem allow a numerical representation, and one looks for an optimal combination of these. The vector analogy in itself lends a geometrical framework within which one may elaborate the picture with concepts allowing a visualization like the Pareto surface.

The multiple criteria decision problem takes us to a very fundamental mathematical property. The set of real numbers as well as its subsets have a natural ordering, but this is no more the case in a space of two or more dimensions. Comparing vectors means that some kind of decision has to be made. There are some evident solutions. One can compare the components one at a time, i.e. use a lexicographic ordering. However, the order of the components has to be decided. Or one can attach a norm to each vector. This is the way mathematics often works, but in practical optimization the normalization of the vector's components can be quite arbitrary. So say some kind of Pythagorean norm may be nonsensical.

The methods in making different components of the decision vector commensurable utilize themselves a number of basic mathematical ideas. A function f transforming a magnitude x to another magnitude $v = f(x)$ more suitable to be a component of the vector to be optimized can be discrete or continuous, and if

continuous assume different convexity and growth properties. And even if linear, the range of f still leaves freedom to subjective choices. Should, for instance, the highest and lowest occurring values of x be carried by f to 1 and 0 or to points reflecting the relative difference of the highest and lowest values of x or should the linear function perform a standardization taking into account the variance of the various observed values of x ?

More subjectivity is introduced by the choice of comparison methods between different vectors. There are ingenious and even mathematically demanding ways of making comparisons, like the eigenvector interpretation of the pairwise preference matrix in the Analytic Hierarchy Process [4], but they cannot hide the fact that there is no objective and always working way to solve the multi-criteria optimization problem. Mathematics can help, but it does not provide the answer.

4.2 Theory of games and making decisions

The theory of games, the basic ideas of which were presented by John von Neumann in the 1930's, deals with conflict, and as such is a possible tool in handling problems of military significance.

As such, game theory utilizes ideas and tools from probability theory and linear optimization. In its more advanced developments, e.g. in multi-person games and differential game theory, a variety of sophisticated mathematical techniques are involved, for instance fixed point theorems from functional analysis.

Game theory is just one example of endeavours to predict future, accommodate decisions to the expectations and try to influence the future. In a more general sense, these considerations involve uncertainties. The mathematician's classical tool in uncertainty handling has been probability. An alternative mathematical description for uncertainty is provided by fuzzy mathematics and logic.

4.3 Queuing theory

Queuing theory¹ has its roots in telephone communication. There are plenty of practical problems, say within logistics, which can be viewed as queuing problems.

Queuing theory is a well-developed branch of stochastic analysis and as such can be considered to be out of scope of our considerations. On the other hand, one finds a chapter devoted to the basics of queuing theory in most operations research text-books. The stochastic process approach typical to queuing problems has applications in different settings where one has to deal with phenomena with uncertain occurrence intervals, say in reliability contexts.

As such, queuing theory falls into the category of operations research methods with the principal aim of understanding phenomena, not so directly giving advice in an actual decision situation.

¹ Scientists in the field prefer the spelling *queueing*.

4.4 Theory of combat

From the military operations research point of view, theory of combat is one of the main subfields of operations research. Also here we see the basic motivation to be the attempt to understand and quantify the complex phenomenon of a battle. Of course this understanding, if reached, can guide the military commander in his preparation for a battle and the military strategist in anticipating future and trying to change it into a direction favourable to his side.

The bulk of the extensive literature on the theory of combat revolves around the concept of attrition and the Lanchester attrition equations

$$y'(t) = -ax(t), \quad x'(t) = -bx(t) \quad (3)$$

or some of their variations. In a way the Lanchester equations assume the position of axioms of a theory resembling the status of Newton's laws in classical mechanics. (There have been attempts to indeed base the theory of combat on an axiomatic foundation [1].) The resemblance is superficial. While centuries of observations verify the basic assumptions of natural sciences, combat is a complicated mixture of events which simply cannot be described by just a couple of parameters and initial conditions, although the equations like (3) or some equivalent systems have a definite solution.

Still, without a clear alternative, study of combat equations can lead to valuable insights and decisions more qualified than decisions based on sheer intuition or things learned from the past. Thus the mathematical ideas and machinery used in the description and predicting attrition should be known to military thinkers.

A differential equation is distinguished from other kinds of equations that need to be solved by the appearance of derivatives of the unknown function. The word derivative is a cover-up for rate of change or velocity. Lanchester equations in a way lead to the very centre of mathematical analysis, the branch of mathematics whose modern content has grown out of the need to quantify change. The equations (3), describing the so called Lanchester's square law, are solved by transforming them to a single second order differential equation with constant coefficients, a type of equations customarily treated in elementary courses of differential equations. The solution which then appears in terms of exponential or hyperbolic functions is not very transparent. However, if we forget the dependence on time and concentrate just on the relationship of the two forces, x and y , we obtain a simple and informative solution which can be described and visualized as a branch of a hyperbola and from which immediate consequences of the outcome of the battle can easily be made.

This approach to a solution of Lanchester's basic attrition equation is a manifestation of the fact that knowing rates of change of variables (forces) in terms of other variables (time), enables one to compute the change of rate one force variable with respect to another force variable. – Of course it is not always this simple. Bringing more details to the model forces one to consider a number of differential

equations. To handle the mathematics, one then has to resort to equations with vector variables and matrix coefficients, and the equations themselves need to be solved numerically by approximate methods.

5 Is there a need of mathematics in the military?

As such a positive answer to the question is self-evident. Lots of technologies that are vital to defence involve deep and complicated mathematics. But we are now considering the question in the context of operations research.

5.1 Decision maker – operational analyst – mathematician

Planning for war is in many ways a challenging task. If we put aside the obvious mental obstacle which lies in the simple fact that war is an undesirable thing, we still face the reality of planning for something of which experience is defective: since previous wars, technology has evolved, the opponent's tactics is different and one cannot have a complete trust in one's own resources, be they technologies or men. The science of war and the predictions or guidelines for action it provides are not of a very exact nature. Quantitative estimates of the possible outcomes of the decisions made by military leaders may be incorrect but if made properly, they are not intolerably incorrect. A military leader endowed with an ability to make use of operational analytic tools and an operational analytic approach is – probably at least – more successful in the average than one who makes his decisions purely on qualitative evidence and intuition.

Another aspect of the relationship between operations research and military lies in the situation, lucky of course, that the activities of most military organizations are restricted to the preparatory phases of conflict, not actually waging wars. As such, military organizations have to deal with problems not much unlike those of any large governmental or commercial organization. The activities are directed by military leaders who then should have competence comparable to leaders in these non-military organizations.

5.2 Mathematics in officer education

Reflecting on the role operational research and mathematics might play in the planning and execution of military operations (understood in the very broad sense which includes material, educational and mental preparation on every level) one is faced with a question similar to any other field of expertise. Decisions should be based on factual information and a deep understanding of its meaning. This would require knowledge so wide-ranging that it cannot be reached by a single person. So the mathematician as an expert of some of the tools used by the operational analyst most likely will always be very much distanced from the decision maker and he cannot expect the decision maker – officer – to be willing to meet and communicate with him in his – the mathematician's – terms.

There is a long-standing tradition to include some basic mathematics in the education of officers. The motivation for this has been the obvious use of mathematics in describing and understanding the natural processes involved in humans, weapons and communications. What would change if one tried to tailor the mathematics curriculum of officer education taking into account the requirements of operations research? Looking at the mathematics in combat analysis and optimization, one still would need a substantial dose of the basics of classical analysis and linear algebra. Simulation and game theory rely on probability, traditionally a basic ingredient of the mathematics of officer education.

The traditional method of presenting mathematics is to start the construction from the basement level (which has no windows). To try an approach in which motivation is provided by first showing the needs of mathematics may seem attractive. At least the insight of the existence, possibilities and ramifications of the chain decision – analysis – mathematics ought to be carried into the consciousness of present and future decision makers. Mathematics is not just a tool of the engineer, it pervades a much wider part of the field on which military decisions grow.

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