Engel, Arthur:

Problem-Solving Strategies

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Mathematical competitions are undoubtedly the most popular extracurricular activity in mathematics. A survey performed in early nineties by the World Federation of National Mathematical Competitions indicates that at least 4 million students participate annually in various pre-university level mathematical competitions. The competitions range from local and regional all the way to the International Mathematical Olympiad, IMO, which has grown from a modest beginning of seven Eastern European participating countries to a worldwide event bringing together more than 400 students from some 80 countries.

The competition problems fall roughly in two categories. In certain mass competitions, the questions are multiple-choice, numerous, but not very difficult, which makes the competition accessible to students of different abilities and evaluation simple. But in the more serious and demanding competitions, the participants are asked to present the solutions, most often in the form of a proof, of a few problems admitting no straightforward cook-book approach. This reflects the philosophy according to which the most enjoyable moments mathematics can give to its pursuer are the ones in which one finally gets the upper hand of a hard-resisting problem.

Mathematical competitions are almost always linked with school mathematics. So the mathematical content of mathematical competitions should theoretically reflect school curricula, but in practice the mathematics represented in these events is a special mixture of certain branches of mathematics thought to require not too much special knowledge and present a variety of possibilities of posing fresh, demanding but manageable questions. The core of "olympiad mathematics" consists of classical Euclidean geometry, mostly in the plane, elementary number theory, algebra of polynomials, basic inequalities, functional equations and a loosely bounded area which could be called combinatorics. On the other hand e.g. calculus or linear algebra and most often probability are not included. So for the most part olympiad mathematics is not taught to anyone neither at school nor at university.

To compete successfully in mathematics one has to be a born genius (which is uncommon) or one has to be quite good and willing to become familiar with the kind of mathematics one meets at competitions. Arthur Engel's book is intended to the latter category. It is an outgrowth of the author's highly successful work as the chief coach of the West German IMO team in the 80's. From the reader it presupposes a considerable mathematical maturity. It is not the book from which one learns the first steps needed to become an olympiad medallist.

The book contains 165 numbered examples with immediate solutions and 1120 numbered excercises, divided into 14 chapters. The solutions of the excercises of any chapter are collected at the end of the same chapter, this makes the book more comfortable to use than the usual problem book arrangement in which the solutions form a separate section at the end. The existence of solutions makes Engel's book superior to the somewhat similar book by L. Larson, *Problem-solving through Problems* (Springer, 1983), in which most problems are presented without solution. The solutions often give just the outline or basic idea of the solution. The competition trainee or trainer should bear in mind that what is required in competitions usually is a more detailed exposition of the solution. Without any explanation, in some chapters (number theory, geometry) the solutions of a few of the last problems of the set are missing.

The division into chapters is based partly on possible strategies, partly on the mathematical content of the problems. The material is not presented very systematically. For instance, right in the first example of the book one utilizes the properties of arithmetic and geometric mean, actually presented in the inequality chapter, and a majorization principle of sequences, not encountered in a usual school curriculum. Some of the chapters, like the ones about number theory, inequalities or polynomials contain a brief synopsis of the main results usually encountered in olympiad problems. The chapter on box principle contains a rather deep excursion into Ramsey theory.

The sources of the problems are only sporadically given. In fact, the author gives no source if he has known the problem for over a quarter of a century. The list of abbreviations shows that in addition to IMO problems the problems of various competitions in the Soviet Union and its followers have been a rich source.

Some of the chapters are devoted to principles which can be applied to a variety of problems. The first chapter deals with what the author calls the Invariance Principle: if something changes, look at what remains invariant. A sample problem: Each of the numbers a_i, $i = 1, \dots, n$ is either 1 or -1. Moreover, $a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0$. Show that n is a multiple of 4. Solution: change the sign of any a_i then exactly 4 terms in the sum change their sign. The sum either remains the same or is increased or decreased by 4 or 8. Change all negative a_i's into positive, one by one. After each step the sum remains divisible by 4, and at the final stage the sum equals n. We are done. Chapter three is centered around the extremal principle: when solving a problem, it is often useful to consider elements which are in some respect extremal. A classic example is the well-known problem in which one has to prove that if a set S of points in the plane is such that every line containing two points of S contains a third one, then S is contained in a line. Assuming this is not the case, one can consider all pairs (P, L), where $P \in S$ and L is a line containing two points of S, $P \notin L$. Looking at the pair for which the distance of P and L is minimal one easily gets a contradiction. – Engel's examples and problems show that the possibilities of seeing change or invariance as well as extremality in problems are endless.

The principles are very general ones, certainly not rules to be applied mechanically or blindly.

A field almost always present in competitions is classical Euclidean geometry. It is almost an article of faith of the IMO juries, for instance, that two of the six problems in the IMO set have to be in geometry. In Engel's book, the chapter on geometry is the longest. It is divided into subsections dealing with vector methods, complex numbers and transformations. Classical Euclidean geometry is mostly represented by excercises, 172 altogether. The brute force method of writing down equations of all relevant lines and circles and trying to handle the situation by algebra, shunned by most competition juries but often leading to solutions, even if nonelegant, is not dealt with.

In the preface, the author hints at the possibility of misprints and minor slips. There are indeed some, most of them harmless. In some instances the reader might be surprised by a lack of explanatory background. Starting a paragraph by "The next problem was submitted in 1988 by the FRG" might sound nonsensical to someone not familiar with mechanisms of the IMO, where the competition problems are selected from suggestions by participating countries. So the problem in question was one suggested by the Federal Republic of Germany for the IMO in 1988. The subsequent discussion of the problem is quite interesting: it was considered imposible by a number of first-rate mathematicians, but was eventually solved by 11 students in the competition.

In spite of its name, the book does not constitute a comprehensive exposition of explicit heuristic strategies, like the ones presented in Polya's classic books. The basic question, how to find the strategy that works, has no easy answer. The strategies are learned implicitly by doing, by massive problem solving (the wording Engel uses in connection of number theoretic problems). A problem would not be a problem, should there exist a mechanical rule by which one could select the tools useful in the solution. There is no doubt that Engel's book will be the basic source for IMO and similar training. And the first thing for committees or individuals wishing to set problems in competitions to do is to check whether their candidates can be found in the book. In the hands of a competent instructor, the book could also serve as the text of a problem course which should be included in teacher training. My pessimistic view is, however, that the average teacher will find the book too hard.

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